The Probability Tutoring Book

AN INTUITIVE COURSE
FOR ENGINEERS AND SCIENTISTS
(AND EVERYONE ELSE!)

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IEEE
PRESS

The Institute of Electrical and Electronics Engineers, Inc., New York

IEEE PRESS
PO Box 1331, 445 Hoes Lane
Piscataway, New Jersey 08855-1331

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-87942-293-0

IEEE Order Number: PC0288-1

Library of Congress Cataloging-in-Publication Data

Ash, Carol (date)
The probability tutoring book : an intuitive course for engineers and scientists (and everyone else!) / Carol Ash.
p. cm.
"IEEE order number: PC0288-1"–T.p. verso.
Includes index.
ISBN 0-87942-293-9
1. Engineering mathematics. 2. Probabilities. I. Title.
TA340.A75 1993
620'.001'51—dc20 92-53183
CIP

Contents

The list of outcomes is countably infinite, so finding the probability that it takes at most 10 tosses to get the first head is a discrete problem.

Here's an example of a continuous problem:

John will be passing the corner of Main and First at some time between 1:00 and 2:00.
Mary will be passing the same corner between 1:30 and 2:00.
Each agrees to wait 5 minutes for the other.
Find the probability that they meet.

There is an infinite number of possible arrival times for John (all the numbers between 1 and 2), and it is an "uncountably" infinite number since there is no way to list all the times and label then 1st, 2nd, 3rd, ... Similarly for Mary. So the problem is continuous.

Discrete probability has more charm, but if you are in engineering, continuous probability will most likely be more useful for you. Discrete probability has no special prerequisite—high school algebra is enough. For continuous probability, you'll need integral calculus. Techniques of integration are not important—a computer can always do the hard integrals—but you will have to remember how to set up a double integral; a review of double integrals is included before they're used in Chapter 5.

The text begins with discrete probability in Chapters 1–3. The rest of the book, Chapters 4–9, covers continuous probability with occasional flashbacks to the discrete case. Discrete and continuous probability have certain basic ideas in common but in practice they will seem quite different. I hope you enjoy them both.

CHAPTER 1

Basic Probability

SECTION 1-1 PROBABILITY SPACES

We want to answer the questions "What are probabilities?" and "How does an event get a probability?"

Sample Space of an Experiment

A sample space corresponding to an experiment is a set of outcomes such that exactly one of the outcomes occurs when the experiment is performed. The sample space is often called the *universe*, and the outcomes are called *points* in the sample space.

There is more than one way to view an experiment, so an experiment can have more than one associated sample space. For example, suppose you draw one card from a deck. Here are some sample spaces.

sample space 1 (the most popular) The space consists of 52 outcomes, 1 for each card in the deck.

sample space 2 This space consists of just 2 outcomes, black and red.

sample space 3 This space consists of 13 outcomes, namely, 2, 3, 4, . . . , 10, J, Q, K, A.

sample space 4 This space consists of 2 outcomes, picture and non-picture.

Any outcome or collection of outcomes in a sample space is called an *event*, including the *null* (empty) set of outcomes and the set of *all* outcomes.

In the first sample space, "black" is an event (consisting of 26 points). It is also an event in sample space 2 (consisting of 1 point). It is *not* an event in sample spaces 3 and 4, so these spaces are not useful if you are interested in the outcome black.

Similarly, "king" is an event in sample spaces 1 and 3 but not in 2 and 4.

Probability Spaces

Consider a sample space with n points.

Probabilities are numbers assigned to events satisfying the following rules.

(1) Each outcome is assigned a non-negative probability such that the sum of the n probabilities is 1.

This axiom corresponds to our intuitive understanding of probabilities in real life. The weather reporter never predicts a negative chance of snow, and the chance of snow plus the chance of rain plus the chance of dry should be 100%, that is, 1.

(2) If A is an event and $P(A)$ denotes the probability of A, then

$$P(A) = \text{ sum of the probabilities of the outcomes in the event } A$$

A sample space together with an assignment of probabilities to events is called a *probability space*. Note that probabilities are always between 0 and 1.

Figure 1 shows a probability space with six outcomes a, b, c, d, e, f and their respective probabilities. The indicated event B contains the three outcomes d, e, f and

$$P(B) = .1 + .2 + .3 = .6$$

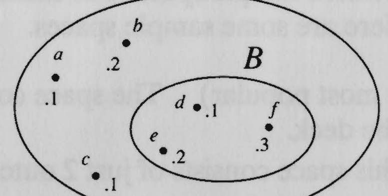


Figure 1

Probabilities may be initially assigned to outcomes any way you like, as long as (1) is satisfied. Then probabilities of events are determined by (2).

To make our probabilities useful, we try to assign initial probabilities to make a "good" model for the experiment.

Suppose you put slips of paper labeled a, b, c, d, e, f in a bag, shake it thoroughly, put on a blindfold, and draw one out; that is, you pick one of a, b, c, d, e, f at random. The appropriate model should have the six outcomes equally likely, so instead of the probabilities in Fig. 1, you should assign each outcome the probability 1/6. Then

$$P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{\text{number of outcomes in the event}}{\text{total number of outcomes}}$$

This suggests an important special case. Suppose a sample space has n points and we make the special assignment

$$(1') \qquad P(\text{each outcome}) = \frac{1}{n}$$

(Note that axiom (1) is satisfied as required.) Then the outcomes are equally likely. In this case it follows that

$$(2') \qquad \boxed{\begin{aligned} P(\text{event}) &= \frac{\text{number of outcomes in the event}}{\text{total number of outcomes}} \\ &= \frac{\text{favorable outcomes}}{\text{total outcomes}} \end{aligned}}$$

Use (1′) and (2′) if an experiment is "fair," in particular if an outcome is picked at random.

For the problems in this book, you may assume that *dice and coins and decks of cards are fair* unless specifically stated otherwise. If an experiment is not fair (e.g., toss a biased coin), you will be given the initial probabilities. In real life you might select the initial probabilities by playing with the coin: If you toss it many times and 63% of the tosses are heads, then it is reasonable to make the initial assignment P(heads) = .63.

How do you decide if your mathematical model (the probability space) is a good one? Suppose you assign initial probabilities so that $P(\text{event } B)$ turns out to be .37. If many people each perform the experiment many times and "most" of the people find that B happens "close to" 37% of the time, then you have a good model.

Example 1 (a fair deck)

Draw one card from a deck. Consider sample space 1 (containing 52 points). In the absence of any special information to the contrary we always choose to assign probabilities using (1′) and (2′). Then

$$P(\text{ace of spades}) = \frac{1}{52}$$

$$P(\text{ace}) = \frac{\text{favorable}}{\text{total}} = \frac{4}{52} = \frac{1}{13}$$

$$P(\text{card}) \geq 10) = \frac{20}{52}$$

> In this text, the ace is considered to be a picture, along with jack, queen, king.
>
> Unless otherwise stated, the ace is high, so that
>
> $$\text{ace} > \text{king} > \text{queen} > \text{jack} > 10$$

Example 2 (a biased deck)

Suppose that the 16 pictures in sample space 1 are assigned probability 1/32 and the 36 non-pictures are assigned prob 1/72. (Note that the sum of the probs is 1, as required.) This corresponds to a deck in which the pictures are more likely to be drawn (maybe the pictures are thicker than the non-pictures). Then

$$P(\text{ace of spades}) = \frac{1}{32}$$

$$P(\text{ace}) = \text{sum of probs of the aces}$$

$$= \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} = \frac{4}{32}$$

$$P(\text{card} \geq 10) = \text{probs of the 10's} + \text{probs of the pictures}$$

$$= 4 \cdot \frac{1}{72} + 16 \cdot \frac{1}{32}$$

$$P(\text{card} \geq 2) = \text{sum of all the probs} = 1$$

$P(\text{red spade}) = 0$ since the event red spade contains no outcomes

Impossible Events and Sure Events

Probabilities are always between 0 and 1.

A *sure* event is one that contains *all* points in the sample space (e.g., card ≥ 2). The probability of a sure event is 1.

An *impossible* event is an event that contains *no* points in the sample space (e.g., red spade). The probability of an impossible event is 0.

The converses are not necessarily true: There are possible events that have probability 0 and non-sure events with probability 1. (See the footnote in Section 2.1 and see (8) in Section 4.1.)

Complementary (Opposite) Events

If A is an event, then its complement $\bar{A}$ is the set of outcomes *not* in A.

For example, if A is red card, then $\bar{A}$ is black card; if $\bar{A}$ is king, then A is non-king.

It follows from (1) and (2) that in any probability space

(3) $$\boxed{P(A) = 1 - P(\bar{A})}$$

Example 3 (tossing two fair dice)

Let's find the probability of getting an 8 (as a sum) when you toss a pair of fair dice.

The most useful sample space consists of the following 36 points; think of one die as red and the other as blue—each point indicates the face value of the red die and the face value of the blue die.

(4)

1, 1	2,	3, 1	4, 1	5, 1	6, 1
1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

There are five outcomes favorable to 8, namely, (2, 6), (6, 2), (5, 3), (3, 5), (4, 4). So

$$P(8) = \frac{5}{36}$$

Example 4

Toss two dice. Find the probability that they show *different* values, for example, (4, 6) and (2, 3) but *not* (2, 2).

You can count the favorable outcomes directly, or better still, by (3),

$$P(\text{non-matching dice}) = 1 - P(\text{matching dice}) = 1 - \frac{6}{36} = \frac{5}{6}$$

Problems for Section 1-1

Toss two dice. Find the probability of each of the following events.

1. sum is 7
2. 7 or 11
3. second die > first die
4. at least one of the dice is a 6
5. both dice are ≥ 5
6. at least one die is ≥ 5
7. neither die is over 4
8. both dice are even
9. at least one die is odd

SECTION 1-2 COUNTING TECHNIQUES

In a probability space where the outcomes are equally likely,

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

In order to take advantage of this rule you must be able to find the total number of outcomes in an experiment and the number that are favorable to your event. For the dice problems in the last section, they were easy to find by inspection. But it usually isn't that simple, so we'll first derive some counting procedures before continuing with probability.

The Multiplication Principle

Suppose you have 3 shirts (blue, red, green) and 2 pairs of pants (checked, striped). The problem is to count the total number of outfits.

The tree diagram (Fig. 1) shows all possibilities: there are $2 \times 3 = 6$ outfits.

Instead of drawing the tree, which takes a lot of space, think of filling a pants slot and a shirt slot (Fig. 2). The pants slot can be filled in 2 ways and the shirt slot in 3 ways, and the total number of outfits is the *product* 2×3.

You'll get the same answer if the tree is drawn with 3 shirt branches first, each followed by 2 pants branches. Equivalently, it doesn't matter if you name the first slot pants and the second shirts as in Fig. 2, or vice versa.

(1) If an event takes place in successive stages (slots), decide in how many ways each slot can be filled, and then multiply to get the total number of outcomes

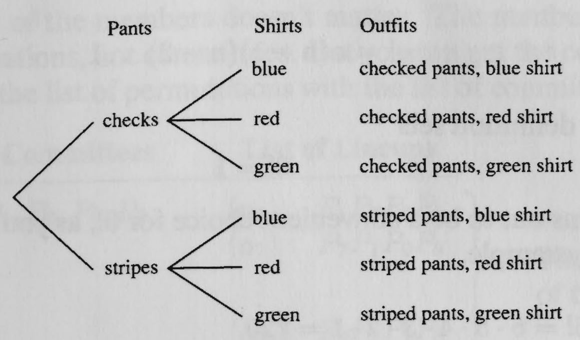


Figure 1

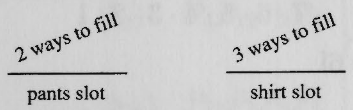


Figure 2 Number of outfits = 3×2

Example 1

The total number of 4-letter words is

$$26 \cdot 26 \cdot 26 \cdot 26$$

(Each spot in the word can be filled in 26 ways.)

Example 2

The total number of 4-letter words that can be formed from 26 different scrabble chips is

$$26 \cdot 25 \cdot 24 \cdot 23$$

(The first spot can be filled in 26 ways, the second in only 25 ways since you have only 25 chips left, etc.)

$7 \times 7 \times 7$ versus $7 \times 6 \times 5$

The answer $7 \times 7 \times 7$ is the number of ways of filling 3 slots from a pool of 7 objects where each object can be used over and over again; this is called sampling *with replacement*. The answer $7 \times 6 \times 5$ is the number of ways of filling 3 slots from a pool of 7 objects where an object chosen for one slot cannot be used again for another slot; this is called sampling *without replacement*.

Review

Let's review the product $n!$ (n factorial), which turns up frequently in applications of the multiplication principle. By definition,

$$n! = n(n-1)(n-2)\dots 1$$

Another definition sets

$$0! = 1$$

(This turns out to be a convenient choice for 0!, as you'll soon see.)
For example,

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

$$\frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 = 720$$

$$5! \times 6 = 6!$$

Permutations (Lineups)

Consider 5 objects $A_1, \dots, A_5$. To count all possible lineups (permutations) such as $A_1A_5A_4A_3A_2$, $A_5A_3A_1A_2A_4$, and so on, think of filling 5 slots, one for each position in the line, and note that once an object has been picked, it can't be picked again. The total number of lineups is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, or 5!.

In general,

> n objects can be permuted in $n!$ ways.

Suppose you want to find the number of permutations of size 5 chosen from the 7 items $A_1, \dots, A_7$, for example, $A_1A_4A_2A_6A_3$, $A_1A_6A_7A_2A_3$. There are 5 places to fill, so the answer is

$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$$

(Some authors will write the answer in the more compact notation $7!/2!$.)

Combinations (Committees)

Now that we've counted permutations, let's try counting committees. I'll illustrate the idea by finding the number of committees of size 4 that can be chosen from the 17 objects $A_1, \dots, A_{17}$.

Note how committees differ from permutations: A_1, A_{17}, A_2, A_{12} is the *same* committee as A_{12}, A_1, A_{17}, A_2, but $A_1A_{17}A_2A_{12}$ and $A_{12}A_1\ A_{17}A_2$ are *different* permutations. Order doesn't count in a committee; it does count in a permutation.

It isn't correct to fill four committee slots and get "answer" $17 \cdot 16 \cdot 15 \cdot 14$ because a committee doesn't have a first member, a second member, and so on—the order of the members doesn't matter. The number $17 \cdot 16 \cdot 15 \cdot 14$ counts permutations, not committees. But you can get the committee answer by comparing the list of permutations with the list of committees.

List of Committees	List of Lineups	
(a) P_1, P_7, P_8, P_9	(a_1) $P_1P_7P_8P_9$	There are 4! of these.
	(a_2) $P_7P_1P_9P_8$	
	$\vdots$	
	(a_{24}) $P_9P_7P_1P_8$	
(b) P_3, P_4, P_{12}, P_6	(b_1) $P_3P_4P_{12}P_6$	There are 4! of these.
	(b_2) $P_4P_{12}P_3P_6$	
	$\vdots$	
	(b_{24}) $P_{12}P_3P_4P_6$	
etc.		

Each committee gives rise to 4! lineups, so

$$\text{number of committees} \times 4! = \text{number of lineups}$$

$$\text{number of committees} = \frac{\text{number of lineups}}{4!} = \frac{17 \cdot 16 \cdot 15 \cdot 14}{3!}$$

$$= \frac{17!}{4!\ 13!}$$

The symbol $\binom{17}{4}$ stands for the number of committees of size 4 from a population of size 17 (it's pronounced 17 on 4 or 17 choose 4). It is also written as $C(17, 4)$. We've just shown that

$$\binom{17}{4} = \frac{17!}{4!\ 13!}$$

Here's the general result:

> The symbol $\binom{n}{r}$, called a *binomial coefficient*, stands for the number of committees of size r that can be chosen from a population of size n or, equivalently, the number of combinations of n things taken r at a time.
>
> Its value is given by
>
> (2) $$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}$$

For example, the number of 4-person committees that can be formed from a group of 10 is

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 10\cdot 3\cdot 7 = 210$$

(Cancel as much as you can before doing any arithmetic.)

Some Properties of $\binom{n}{r}$

(3) $$\boxed{\binom{n}{r} = \binom{n}{n-r}}$$

For example,

$$\binom{17}{4} = \binom{17}{13}$$

This holds because picking a committee of size 4 automatically leaves a committee of 13 leftovers and vice versa so the list of committees of size 4 and the list of committees of size 13 are the same length. Alternatively, $\binom{17}{4}$ and $\binom{17}{13}$ are equal because by the formula in (2) each is $17!/4!\,13!$.

(4) $$\boxed{\binom{n}{1} = \binom{n}{n-1} = n}$$

This follows because there are clearly n committees of size 1 that can be chosen from a population of size n. It also follows from the formula in (2) since

$$\binom{n}{1} = \binom{n}{n-1} = \frac{n!}{1!\,(n-1)!}$$

which cancels down to n.

(5) $$\boxed{\binom{n}{n} = \binom{n}{0} = 1}$$

We have $\binom{n}{n} = 1$ because there is just one way to form a committee of size n from a population of size n. If you try to use the formula in (2), you get

$$\binom{n}{n} = \frac{n!}{n!\,0!}$$

and it will be 1, provided you define $0! = 1$. In other words, you can define 0! any way you like, but it is convenient to call it 1 because then the formula in (2) continues to hold even when r is n or 0.

Once you have $\binom{n}{n} = 1$, it follows from (3) that $\binom{n}{0} = 1$ also. If you want to interpret $\binom{n}{0}$ as the number of committees with no members, look at it like this: There is 1 committee that can be formed with no members, namely, the null (empty) committee.

Example 4

I'll find the probability of getting the queen of spades (denoted Q_S) in a poker hand.

A poker hand is a committee of 5 cards drawn from 52.

The *total* number of poker hands is $\binom{52}{5}$.

Finding *favorable* hands amounts to selecting a committee of size 4 (the rest of the hand) from 51 (the rest of the deck). So there are $\binom{51}{4}$ favorable hands and

$$P(Q_S) = \frac{\binom{51}{4}}{\binom{52}{5}} = \frac{51!}{4!\,47!}\cdot\frac{5!\,47!}{52!} = \frac{5}{52}$$

Example 5

I'll find the probability of *not* getting the queen of spades in poker.

Method 1 (directly) Again, the total number of poker hands is $\binom{52}{5}$. Each favorable hand contains 5 cards chosen from the 51 non-Q_S's. So there are $\binom{51}{5}$ favorable hands, and

$$P(\text{not getting the queen of spades}) = \frac{\binom{51}{5}}{\binom{52}{5}}$$

Method 2 (indirectly)

$$P(\overline{Q_S}) = 1 - P(Q_S) = 1 - \text{ answer to example 4} = 1 - \frac{5}{52}$$

The two answers agree:

$$\text{method 1 answer} = \frac{51!}{5!\,46!}\,\frac{5!\,47!}{52!} = \frac{47}{52} = \text{method 2 answer}$$

$9 \times 8 \times 7$ versus $\binom{9}{3}$

Both count the number of ways in which 3 things can be chosen from a pool of 9. But $9\times8\times7$ corresponds to choosing the 3 things to fill labeled slots (such as president, vice president, secretary), while $\binom{9}{3}$ corresponds to the case where the 3 things are not given different labels or distinguished from one another in any way (such as 3 co-chairs).

From a committee/lineup point of view, $9\times8\times7$ counts lineups of 3 from a pool of 9, while $\binom{9}{3}$ counts committees.

Poker Hands versus Poker Lineups

Consider the probability of getting a poker hand with all hearts.

A poker hand is a committee of cards, and

(6) $$P(\text{all hearts}) = \frac{\binom{13}{5}}{\binom{52}{5}}$$

(For the numerator, pick 5 cards from the 13 hearts.)

If we consider a poker *lineup*, that is, a *lineup* of 5 cards drawn from 52, then

(7) $$P(\text{all hearts}) = \frac{13\cdot12\cdot11\cdot10\cdot9}{52\cdot51\cdot50\cdot49\cdot48}$$

The answers in (6) and (7) agree since

$$\frac{\binom{13}{5}}{\binom{52}{5}} = \frac{13!}{5!\,8!}\cdot\frac{5!\,47!}{52!} = \frac{13\cdot12\cdot11\cdot10\cdot9}{52\cdot51\cdot50\cdot49\cdot48}$$

In (6) the underlying sample space is the set of *unordered* samples of size 5 chosen without replacement from a population of 52. In (7), the sample space is the set of *ordered* samples so that $A_SK_HJ_D2_S3_C$ and $K_HA_SJ_D2_S3_C$ are different outcomes. The probability of all hearts is the *same* in both spaces so it's OK to use poker lineups instead of poker hands. (But (6) seems more natural to most people.)

Warning

In some instances, such as the probability of all hearts in a poker hand, you can use ordered samples or unordered samples as long as you are *consistent* in the numerator and denominator. *If you use order in one place, you must use it in the other place as well.*

Problems for Section 1-2

1. A committee of size 5 is chosen from $A_1, \ldots, A_9$. Find the probability that
 (a) the committee contains A_6
 (b) the committee contains neither A_7 nor A_8

2. Find the probability of a bridge hand (13 cards) with the AKQJ of spades and no other spades.

3. If 2 people are picked from 5 men, 6 women, 7 children, find the probability that they are not both children.

4. Find the probability of a poker hand with
 (a) no hearts
 (b) the ace of spades and the ace of diamonds (other aces allowed also)
 (c) the ace of spades and the ace of diamonds and no other aces

5. Four women check their coats and the coats are later returned at random. Find the probability that
 (a) each woman gets her own coat
 (b) Mary gets her own coat

6. Compute
 (a) $7!/5!$ **(b)** $\binom{7}{5}$ **(c)** $\binom{8}{5}/\binom{4}{3}$ **(d)** $\binom{12345}{1}$ **(e)** $\binom{12345}{0}$ **(f)** $\binom{12345}{12345}$ **(g)** $\binom{12345}{12344}$

7. In a certain computer system you must identify yourself with a password consisting of a single letter or a letter followed by as many as 6 symbols which may be letters or digits, for example, Z, ZZZZZZ6, RUNNER, JIMBO, R2D2. Assuming that any password is as likely to be chosen as any other, what is the probability that John and Mary choose the same password?

8. Consider samples of size 3 chosen from $A_1, \ldots, A_7$.
 (a) Suppose the samples are drawn *with replacement* so that after an item is selected, it is replaced before the next draw (e.g., A_1 can be drawn more than once) and *order counts* (e.g., $A_2A_1A_1$ is different from $A_1A_2A_1$). How many samples are there?
 (b) Suppose the samples are drawn *without replacement* and *order counts*. How many samples are there?
 (c) How many samples are there if the sampling is *without replacement* and *order doesn't count*?
 (d) What is the fourth type of sampling? (Counting in this case is tricky and won't be needed in this course.)

9. Find the probability of a royal flush in poker (a hand with AKQJ 10 all in the same suit).

10. There are 7 churches in a town. Three visitors pick churches at random to attend. Find the probability that

(a) they all choose the same church
(b) they do not all choose the same church
(c) they choose 3 different churches
(d) at least 2 of them choose the same church

11. In the state lottery the winning ticket is decided by drawing 6 different numbers from 1 to 54. The order in which the numbers are drawn is irrelevant so that the draws 2, 54, 46, 37, 1, 6 and 54, 2, 37, 46, 1, 6 are the same.

For $1 you get two tickets, each with your choice of 6 numbers on it. What is the probability that you win the lottery when you spend $1?

SECTION 1-3 COUNTING TECHNIQUES CONTINUED

Here are some examples that are more intricate but can still be done using the multiplication principle and permutation and combination rules from the last section.

Example 1

A committee of size 7 is chosen from 6 men, 7 women, 8 children. Let's find the probability that the committee contains

(a) 2 men, 4 women, 1 child
(b) 2 men

(a) The total number of committees is $\binom{21}{7}$.

For the favorable outcomes, the 2 men can be picked in $\binom{6}{2}$ ways, the 4 women in $\binom{7}{4}$ ways, and the child in 8 ways. (Think of filling three slots: a 2-man subcommittee, a 4-woman subcommittee, and a 1-child subcommittee.) So

$$P(2\text{M}, 4\text{W}, 1\text{C}) = \frac{\binom{6}{2}\binom{7}{4}\cdot 8}{\binom{21}{7}}$$

(b) For the favorable outcomes, the 2 men can be picked in $\binom{6}{2}$ ways, the 5 others in $\binom{15}{5}$ ways. So

$$P(2\text{ men}) = \frac{\binom{6}{2}\binom{15}{5}}{\binom{21}{7}}$$

Example 2

Find the probability that a poker hand contains only one suit.

To count the favorable hands, pick the suit in 4 ways. Then pick the 5 cards from that suit in $\binom{13}{5}$ ways. So

$$P(\text{only one suit}) = \frac{4\cdot\binom{13}{5}}{\binom{52}{5}}$$

Example 3

A box contains 40 white, 50 red, 60 black balls. Pick 20 without replacement. Find the prob of getting

(a) 10 white, 4 red, 6 black
(b) 10 white

(a) The total number of outcomes is $\binom{150}{20}$. For the favorable, pick 10 white out of 40, pick 4 red out of 50, pick 6 black out of 60:

$$P(10\text{W}, 4\text{R}, 6\text{B}) = \frac{\binom{40}{10}\binom{50}{4}\binom{60}{6}}{\binom{150}{20}}$$

(b) For the favorable, pick 10 white out of 40, pick 10 others from the 110 non-white:

$$P(10\text{W}) = \frac{\binom{40}{10}\binom{110}{10}}{\binom{150}{20}}$$

Indistinguishable versus Distinguishable Balls

You may not have realized it, but as we picked committees and used fav/total in example 3, we assumed that balls of the same color could be distinguished from one another; for example, we assumed that balls were named $W_1, \ldots, W_{40}$; $R_1, \ldots, R_{50}$; and $B_1, \ldots, B_{60}$. The probability of getting 10W, 4R, 6B is the same whether or not the balls have names painted on them, so our assumption is not only convenient but legal.

> If an experiment involves n white balls, to find any probabilities we assume the balls are named $W_1, \ldots, W_n$.

Example 4

Consider strings of digits and letters of length 7 without repetition. Find the probability that a string contains 2 digits, 4 consonants, and 1 vowel.

Method 1 For the favorable outcomes, pick 2 digits, 4 consonants, and 1 vowel and then line them up.

$$\text{prob} = \frac{\binom{10}{2}\binom{21}{4}\cdot 5\cdot 7!}{36\cdot 35\cdot 34\cdot 33\cdot 32\cdot 31\cdot 30}$$

Method 2 For the favorable outcomes, pick 2 positions in the string for the digits, 4 places for the consonants, leaving 1 for the vowel. Then fill the spots.

$$\text{prob} = \frac{\binom{7}{2}\binom{5}{4}\cdot 10\cdot 9\cdot 21\cdot 20\cdot 19\cdot 18\cdot 5}{36\cdot 35\cdot 34\cdot 33\cdot 32\cdot 31\cdot 30}$$

Double Counting

I'll find the probability of a poker hand with two pairs.

If I regard a poker hand as a committee, then the total number of hands is $\binom{52}{5}$.

Now I need the number of favorable hands.

To get started, it often helps to write out some favorable outcomes—the steps involved in constructing an outcome may suggest the slots to be filled in the counting process. Here are some poker hands with two pairs:

Hand 1: Q_H, Q_D, J_S, J_H, A_H
Hand 2: $4_C, 4_H, K_D, K_C, 2_H$

First, let's analyze an *incorrect* method.

Each outcome involves a first face (e.g., queen), two suits in that face (e.g., heart and diamond), a second face (e.g., jack), two suits in that face (e.g., spade and heart), and a fifth card not of either face (to avoid a full house). So use these slots:

Step 1. Pick a face value.
Step 2. Pick 2 of the 4 cards in that face.
Step 3. Pick another face value.
Step 4. Pick 2 out of the 4 cards in that face.
Step 5. Pick a fifth card from the 44 of neither face.

WRONG

Filling these 5 slots we get "answer"

$$(*) \qquad 13\cdot\binom{4}{2}\cdot 12\cdot\binom{4}{2}\cdot 44$$

This is wrong because it counts the following as different outcomes when they are really the same hand:

Outcome 1	Outcome 2
Pick queen, hearts and spades.	Pick jack, hearts and clubs.
Pick jack, hearts and clubs.	Pick queen, hearts and spades.
Pick ace of clubs.	Pick ace of clubs.

Step 1 implicitly fills a slot named first face value and step 3 fills a slot named second face value. But in a poker hand, the two faces for the two pairs can't be distinguished as first and second so the slots are illegal.

The "answer" in (∗) counts every outcome twice. (Once you notice this, you can divide by 2 to get the right answer.)

Here's a correct version (from scratch):

Pick a committee of 2 face values for the pairs.
Pick 2 cards from each face value.
Pick a fifth card from the 44 not of either face.
The number of favorable hands is $\binom{13}{2}\binom{4}{2}\binom{4}{2}\cdot 44$.

And the probability of two pairs is

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\cdot 44}{\binom{52}{5}}$$

In general, an "answer" that *counts some outcomes more than once* is referred to as a *double count* (the preceding double count happens to count *every* outcome exactly *twice*). Double counts can be hard to resist.

Warning

> A common mistake is to use, say, $7\cdot 6$ instead of $\binom{7}{2}$, that is, to fill slots implicitly named first thingo and second thingo when you should pick a committee of two things.

Symmetries

Here are some typical symmetries.

Draw cards either with or without replacement.

(1)
$$P(\text{ace on 1st draw}) = P(\text{ace 2nd}) = P(\text{ace 3rd}), \text{ etc.}$$
$$P(\text{ace 3rd, king 10th}) = P(\text{A 1st, K 2nd}) = P(\text{A 2nd, K 1st,}), \text{ etc.}$$

Intuitively, each position in the deck of cards has the same chance of harboring an ace; each *pair* of positions has the same chance of containing an ace and king, and so on.

Similarly, draw with or without replacement from a box containing red, white, and black balls.

$$(2) \qquad \boxed{\begin{aligned} P(\text{RWBW drawn in that order}) &= P(\text{WWRB}) \\ &= P(\text{RBWW}), \text{etc.} \end{aligned}}$$

Whatever the distribution of colors you draw, you're just as likely to get them in one order as another.

It isn't safe to rely on intuition, so I'll do one proof as a justification. I'll show that

$$P(\text{A on 3rd draw, K on 5th draw}) = P(\text{K on 1st, A on 2nd})$$

This is immediate if the drawings are with replacement: Each probability is

$$\frac{4 \cdot 4}{52 \cdot 52}$$

Suppose the drawing is with*out* replacement. Then

$$P(\text{K on 1st, A on 2nd}) = \frac{\text{fav}}{\text{total}} = \frac{4 \cdot 4}{52 \cdot 51}$$

$$P(\text{A on 3rd, K on 5th}) = \frac{\text{fav}}{\text{total}} = \frac{\text{fav}}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

For the fav, there are 5 slots to fill. The 3rd slot can be filled in 4 ways, the 5th slot in 4 ways, and then the other 3 slots in $50 \cdot 49 \cdot 48$ ways. So

$$P(\text{A on 3rd, K on 5th}) = \frac{4 \cdot 4 \cdot 50 \cdot 49 \cdot 48}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{4 \cdot 4}{52 \cdot 51}$$

The "other 3 slots" canceled out, leaving the same answer as P(K 1st, A 2nd), QED.

Example 5

Draw without replacement from a box with 10 white and 5 black balls.

To find the prob of W on the 1st and 4th draws (no information about 2nd and 3rd), take advantage of symmetry and switch to an easier problem:

$$P(\text{W on lst and 4th}) = P(\text{W on 1st and 2nd}) = \frac{10 \cdot 9}{15 \cdot 14}$$

Problems for Section 1-3

1. Find the prob that a poker hand contains
- **(a)** 3 diamonds and 2 hearts
- **(b)** 2 spades, one of which is the ace
- **(c)** 4 black and 1 red
- **(d)** 2 aces
- **(e)** the ace of spades but not the king of spades

2. Three Americans A_1, A_2, A_3, 7 Russians $R_1, \ldots, R_7$ and 8 Germans $G_1, \ldots, G_8$ try to buy concert tickets. Only 5 tickets are left. If the tickets are given out at random, find the prob that
- **(a)** R_3 gets a ticket and so do 2 of the 3 Americans
- **(b)** only 1 of the Germans gets a ticket

3. If 3 people are picked from a group of 4 married couples, what is the prob of not including a pair of spouses?

4. If a 12-symbol string is formed from the 10 digits and 26 letters, repetition not allowed, what is the prob that it contains 3 even digits?

5. Find the prob of getting 3 whites and 2 reds if you draw 11 balls from a box containing 25 white, 30 red, 40 blue, and 50 black.

6. If four people are assigned seats at random in a 7-seat row, what is the prob that they are seated together?

7. Find the prob that a 3-card hand contains 3 of a kind (i.e., 3 of the same value).

8. **(a)** Find the prob that a 4-card hand contains 2 pairs.
- **(b)** Find the prob that a 5-card hand contains a full house (3 of a kind and a pair).

9. Find the prob that a poker hand contains
- **(a)** a flush (5 cards of the same suit)
- **(b)** 4 aces
- **(c)** 4 of a kind (e.g., 4 jacks)
- **(d)** a pair (and nothing better than one pair)

10. Suppose b boys and g girls are lined up at random. Find the prob that there is a girl in the ith spot.

11. Here are some counting problems with proposed answers that double count. Explain *how* they double count (produce specific outcomes which are counted as if they are distinct but are really the same) and then get correct answers.
- **(a)** To count the number of poker hands with 3 of a kind:

  Pick a face value and 3 cards from that value.
  Pick one of the remaining 48 cards not of that value (to avoid 4 of a kind).

Pick one of the remaining 44 not of the first or second value (to avoid 4 of a kind and a full house).

Answer is $13 \cdot \binom{4}{3} \cdot 48 \cdot 44$. WRONG

(b) To count 7-letter words with 3 A's:

Pick a spot for the first A.
Pick a spot for the second A.
Pick a spot for the third A.
Pick each of the remaining 4 places with any of the non-A's.

Answer is $7 \cdot 6 \cdot 5 \cdot 25^4$. WRONG

(c) To count 2-card hands not containing a pair:

Pick any first card.
Pick a second card from the 48 not of the first face value.

Answer is $52 \cdot 48$. WRONG

SECTION 1-4 ORS AND AT LEASTS

OR versus XOR

The word "or" has two different meanings in English. If you've seen Boolean algebra you know that engineers have two different words, OR and XOR, to distinguish the two meanings:

A OR B means A or B or both, called an *inclusive* or.

Similarly, A OR B OR C means one or more of A, B, C (exactly one of A, B, C or any two of A, B, C or all three of A, B, C). On the other hand,

A XOR B means A or B but not both, an *exclusive* or.

In this book, *"or" will always mean the inclusive OR* unless specified otherwise.

In the real world you'll have to decide for yourself which kind is intended. If a lottery announces that any number containing a 6 or a 7 wins, then you win with a 6 or 7 or both; that is, 6 OR 7 wins (inclusive or). On the other hand, if you order a Coke or a 7-Up, you really mean a Coke or a 7-Up but not both; that is, Coke XOR 7-Up.

OR Rule (Principle of Inclusion and Exclusion)

For 2 events,

(1) $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

For 3 events,

(2) $$\begin{aligned} P(A \text{ or } B \text{ or } C) &= P(A) + P(B) + P(C) \\ &\quad - [P(A \,\&\, B) + P(A \,\&\, C) + P(B \,\&\, C)] \\ &\quad + P(A \,\&\, B \,\&\, C) \end{aligned}$$

Here's the general pattern for n events:

(3) $$\begin{aligned} &P(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_n) \\ &= \text{1-at-a-time terms} \\ &\quad -\text{2-at-a-time terms} \\ &\quad +\text{3-at-a-time terms} \\ &\quad -\text{4-at-a-time terms} \\ &\quad \text{etc.} \end{aligned}$$

For example,

$$\begin{aligned} &P(A \text{ or } B \text{ or } C \text{ or } D) \\ &= P(A) + P(B) + P(C) + P(D) \\ &\quad - [P(A \,\&\, B) + P(A \,\&\, C) + P(A \,\&\, D) + P(B \,\&\, C) + P(B \,\&\, D) + P(C \,\&\, D)] \\ &\quad + [P(A \,\&\, B \,\&\, C) + P(A \,\&\, B \,\&\, D) + P(B \,\&\, C \,\&\, D) + P(A \,\&\, C \,\&\, D)] \\ &\quad - P(A \,\&\, B \,\&\, C \,\&\, D) \end{aligned}$$

Proof of (1)

Suppose event A contains the 4 outcomes indicated in Fig. 1 with respective probs $p_1, \ldots, p_4$. And suppose B contains the indicated 3 outcomes.

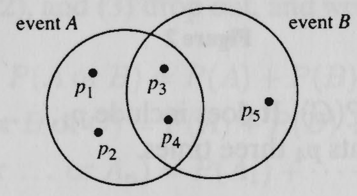


Figure 1

The event "A or B" contains all 5 outcomes in Fig. 1, so

$$P(A \text{ or } B) = p_1 + p_2 + p_3 + p_4 + p_5$$

On the other hand,

$$P(A) + P(B) = p_1 + p_2 + p_3 + p_4 \quad + \quad p_3 + p_4 + p_5$$

This is *not* the same as $P(A \text{ or } B)$ because it counts the probs p_3 and p_4 *twice*. We *do* want to count them since this is an inclusive or, but we don't want to count them twice. So to get $P(A \text{ or } B)$, start with $P(A) + P(B)$ and then subtract the probs in the intersection of A and B, that is, subtract $P(A \text{ and } B)$ as in (1).

Warning

The "or" in rule (1) is *inclusive*; it means A or B *or both*. We subtract away $P(A\&B)$ not because we want to throw away the both's but because we don't want to count them twice.

In other words,

$$P(A \text{ or } B) = P(A \text{ or } B \textit{ or both}) = P(A) + P(B) - P(A\&B)$$

Proof of (2)

Suppose A contains the 5 outcomes in Fig. 2 with indicated probs, B contains the indicated 6 outcomes, and C contains the indicated 4 outcomes. Then

$$P(A \text{ or } B \text{ or } C) = p_1 + \cdots + p_9$$

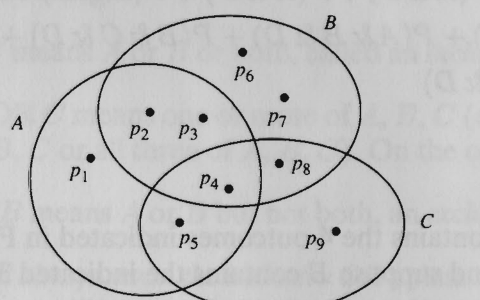


Figure 2

Look at $P(A) + P(B) + P(C)$. It does include $p_1, \ldots, p_9$, but it counts p_2, p_3, p_5, p_8 twice each and counts p_4 three times.

If we subtract away

$$\underbrace{P(A\&B)}_{p_2+p_3+p_4}, \quad \underbrace{P(A\&C)}_{p_4+p_5}, \quad \underbrace{P(B\&C)}_{p_4+p_8}$$

then p_2, p_3, p_5, p_8 will be counted once each, not twice, but now p_4 isn't counted at all. So formula (2) adds $P(A\&B\&C)$ back in to include p_4 again.

Example 1

I'll find the probability that a bridge hand (13 cards) contains 4 aces or 4 kings.

By the OR rule,

$$P(4 \text{ aces or } 4 \text{ kings}) = P(4 \text{ aces}) + P(4 \text{ kings}) - P(4 \text{ aces and } 4 \text{ kings})$$

The total number of hands is $\binom{52}{13}$.

When you count the number of ways of getting 4 aces, don't think about kings at all (the hand may or may not include 4 kings—you don't care). The other 9 cards can be picked from the 48 non-aces so there are $\binom{48}{9}$ hands with 4 aces. Similarly, for outcomes favorable to 4 kings, pick the other 9 cards from the 48 non-kings. And for the outcomes favorable to the event 4 kings and 4 aces, pick the other 5 cards from the 44 remaining cards. So

$$P(4 \text{ aces or } 4 \text{ kings}) = \frac{\binom{48}{9}}{\binom{52}{13}} + \frac{\binom{48}{9}}{\binom{52}{13}} - \frac{\binom{44}{5}}{\binom{52}{13}}$$

Unions and Intersections

Many books use the union symbol instead of "or" and the intersection symbol instead of "and" so that the OR rule for two events looks like this:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

I use "or" and "and" because they seem more natural. Most people would refer to 3 aces *or* 3 kings rather than to the *union* of 3-ace hands and 3-king hands.

Mutually Exclusive (Disjoint) Events

Suppose events A, B, C, D are mutually exclusive, meaning that no two can happen simultaneously (A, B, C, D have no outcomes in common). Then all the "and" terms in (1), (2), and (3) drop out, and we have

(1′) $$P(A \text{ or } B) = P(A) + P(B)$$

(2′) $$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$$

(3′) $$P(A_1 \text{ or } \ldots \text{ or } A_n) = P(A_1) + \cdots + P(A_n)$$

Example 2

Consider poker hands containing all spades or all hearts. The events "all spades" and "all hearts" are mutually exclusive since they can't happen simultaneously. So by (1′),

$$P(\text{all spades or all hearts}) = P(\text{all spades}) + P(\text{all hearts})$$

$$= \frac{\binom{13}{5}}{\binom{52}{5}} + \frac{\binom{13}{5}}{\binom{52}{5}}$$

Warning

$P(A \text{ or } B)$ is not $P(A) + P(B)$ *unless* A and B are mutually exclusive. If they are not, *don't forget to subtract* $P(A\&B)$.

At Least One

To illustrate the general idea I'll find the probability that a poker hand contains at least one ace.

Method 1 (the best for this particular example)

$$P(\text{at least one ace}) = 1 - P(\text{no aces}) = 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

Method 2

$$P(\text{at least one ace}) = P(1\text{A or } 2\text{A or } 3\text{A or } 4\text{A})$$

The events one ace (meaning *exactly* one ace), two aces, three aces, four aces are mutually exclusive, so we can use the abbreviated OR rule.

$$P(\text{at least one ace}) = P(1\text{A}) + P(2\text{A}) + P(3\text{A}) + P(4\text{A})$$

$$= \frac{4\binom{48}{4} + \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + 48}{\binom{52}{5}}$$

Method 3

$$P(\text{at least one ace}) = P(A_S) + P(A_H) + P(A_C) + P(A_D)$$
$$- [P(A_S \,\&\, A_H) + \text{ other 2-at-a-time terms}]$$
$$+ [P(A_S \,\&\, A_H \,\&\, A_C) + \text{ other 3-at-a-time terms}]$$
$$- P(A_S \,\&\, A_H \,\&\, A_C \,\&\, A_D)$$

This long expansion isn't as bad as it looks.

The first bracket contains 4 terms all having the same value, namely, $\binom{51}{4}/\binom{52}{5}$.

The second bracket contains $\binom{4}{2}$ terms; all have the value $\binom{50}{3}/\binom{52}{5}$.

The third bracket contains $\binom{4}{3}$ terms, each with the value $\binom{49}{2}/\binom{52}{5}$.

So

$$P(\text{at least one ace}) = \frac{4\binom{51}{4} - \binom{4}{2}\binom{50}{3} + \binom{4}{3}\binom{49}{2} - 48}{\binom{52}{5}}$$

In this example, method 1 was best, but you'll see examples favoring each of the other methods.

Warning

Here is a wrong way to find the probability of at least one ace in a poker hand. The denominator is $\binom{52}{5}$ (right so far). For the numerator:

WRONG
Pick 1 ace to be sure of getting at least one.
Pick 4 more cards from the remaining 51.
So the numerator is $4\binom{51}{4}$.

The numerator is wrong because it counts the following as different outcomes when they are the same:

Outcome 1	Outcome 2
Pick the A_S as the sure ace. Then pick $Q_H, A_H, 2_D, 6_C$.	Pick the A_H as the sure ace. Then pick $Q_H, A_S, 2_D, 6_C$.

So the numerator *double counts.*

> Don't try to count at least n thingos by presetting n thingos to be sure and then going on from there. It just won't work.

Example 3

Here's how to find the prob that a bridge hand contains at most 2 spades.

Method 1

$$P(\text{at most 2 spades}) = 1 - P(3\text{S or } 4\text{S or } \ldots \text{ or } 13\text{S})$$
$$= 1 - [P(3\text{S}) + P(4\text{S}) + \cdots + P(13\text{S})]$$

OK, but too slow!

Method 2

$$P(\text{at most 2 spades}) = P(\text{no S}) + P(1\text{S}) + P(2\text{S})$$

$$= \frac{\binom{39}{13} + 13\binom{39}{12} + \binom{13}{2}\binom{39}{11}}{\binom{52}{13}}$$

Exactlies Combined with At Leasts

I'll find the probability of a poker hand with 2 spades and at least 1 heart.

Method 1

Use a variation of the rule

$$P(\text{at least 1 heart}) = 1 - P(\text{no hearts})$$

to get

$$P(\text{2 spades and at least 1 heart}) = P(2\text{S}) - P(2\text{S and no H})$$

$$= \frac{\binom{13}{2}\binom{39}{3} - \binom{13}{2}\binom{26}{3}}{\binom{52}{5}}$$

Method 2

$$P(\text{2 spades and at least 1 heart})$$
$$= P(2\text{S and H}) + P(2\text{S and 2H}) + P(2\text{S and 3H})$$

$$= \frac{\binom{13}{2} \cdot 13 \cdot \binom{26}{2} + \binom{13}{2}\binom{13}{2} \cdot 26 + \binom{13}{2}\binom{13}{3}}{\binom{52}{5}}$$

At Least One of Each

Form committees of 6 from a population of 10 Americans, 7 Russians, and 5 Germans.

$$P(\text{at least 1 of each European nationality})$$
$$= P(\text{at least one R and at least one G})$$
$$= 1 - P(\text{no R or no G})$$
$$= 1 - [P(\text{no R}) + P(\text{no G}) - P(\text{no R and no G})]$$
$$= 1 - \frac{\binom{15}{6} + \binom{17}{6} - \binom{10}{6}}{\binom{22}{6}}$$

Warning

The complement of

at least 1 of each European nationality

is *not* no Europeans; that is, the complement is *not*

no R *and* no G

Rather, the complement is

no R *or* no G

Some Basic Pairs of Complementary Events

Here is a brief list of some complementary events. (You should understand the logic of the list rather than force yourself to memorize it.)

Event	Complement
A or B	not A and not B
A and B	not A or not B
at least 1 king	no kings
at least 1 king and at least 1 queen	no kings or no queens
at least 1 king or at least 1 queen	no kings and no queens
at least 1 of each suit	no S or no H or no C or no D

Problems for Section 1-4

1. Find the prob that a poker hands contains
   (a) 2 aces or 2 kings
   (b) 3 aces or 3 kings
   (c) the ace or king of spades

2. A box contains 10 white balls, 20 reds, and 30 greens. Draw 5 without replacement. Find the prob that
   (a) the sample contains 3 white or 2 red or 5 green
   (b) all 5 are the same color

3. Consider computing $P(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_8)$.
   (a) Eventually, you have to subtract away the 2-at-a-time terms such as $P(A_5 \,\&\, A_3)$. How many of these terms are there?
   (b) Eventually, you will have to add in the 3-at-a-time terms such as $P(A_1 \,\&\, A_4 \,\&\, A_5)$. How many are there?

4. A jury pool consists of 25 women and 17 men. Among the men, 2 are Hispanic, and among the women, 3 are Hispanic. If a jury of 12 people is picked at random what is the prob that it

(a) contains no women or no Hispanics
(b) contains no women and no Hispanics

5. Find the prob that a poker hand contains the jack of spades XOR the queen of spades (the jack or queen but not both).

6. Find the prob that a poker hand contains

(a) at least 1 spade
(b) at least 3 spades
(c) at most 2 aces
(d) 4 pictures including at least 2 aces

7. Find the prob that a bridge hand contains

(a) at least one royal flush (AKQJ 10 in the same suit)
(b) at least one 4-of-a-kind (e.g., 4 kings)

8. A hundred people including the Smith family (John, Mary, Bill, Henry) buy a lottery ticket apiece. Three winning tickets will be drawn without replacement. Find the prob that the Smith family ends up happy. (Find three methods if you can, for practice.)

9. Find the prob that a 4-person committee chosen from 6 men, 7 women, and 5 children contains

(a) 1 woman
(b) at least 1 woman
(c) at most 1 woman
(d) at least 1 of each category
(e) no women and at least 1 man

10. There are 50 states and 2 senators from each state. Find the prob that a committee of 15 senators contains

(a) at least 1 from each of Hawaii, Massachusetts, and Pennsylvania
(b) at least 1 from the three-state region composed of Hawaii, Massachusetts, and Pennsylvania
(c) 1 from Hawaii and at least 1 from Massachusetts

11. (the game of rencontre—the matching game) Seven husbands $H_1, \ldots, H_7$ and their wives $W_1, \ldots, W_7$ are matched up at random to form 7 new coed couples.

(a) Find the prob that H_3 is matched with his own wife.
(b) Find the prob that H_2, H_5, and H_7 are all matched with their own wives.

(c) Find the prob that at least one husband is matched with his wife and simplify to get a pretty answer.
Suggestion: The only feasible method is to use

$$P(\text{at least one match}) = P(H_1 \text{ is matched or } H_2 \text{ or } \ldots \text{ or } H_7)$$

(d) Find the prob that no husband is matched with his wife.

Review

Here's one way to draw the graph of the inequality $x - y < 2$. First, draw the graph of the equation $x - y = 2$, a line. The line divides the plane into two regions (Fig. 1). One of the regions corresponds to $x - y < 2$ and the other to $x - y > 2$. To decide which region goes with which inequality, test a point. Point $(100, 0)$ for instance is in region 2, and it satisfies the inequality $x - y > 2$. So it is region 1 that must be the graph of $x - y < 2$.

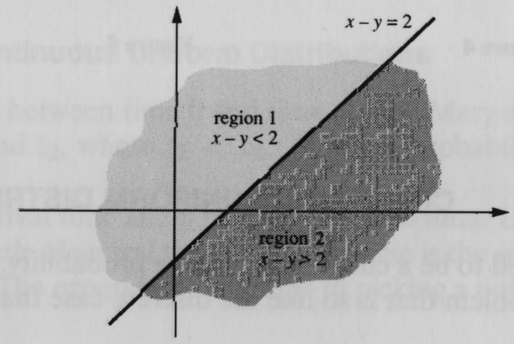


Figure 1

Another way to graph inequalities is to solve for y. The graph of

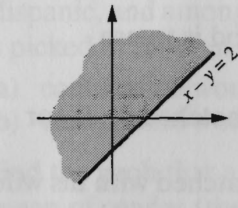
$$y < x - 2$$

is the region *below* line $y = x - 2$ and the graph of

$$y > x - 2$$

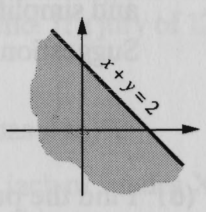
is the region *above* line $y = x - 2$.

Here are some more graphs of inequalities.



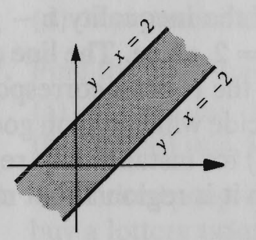
$x - y < 2$

Figure 2



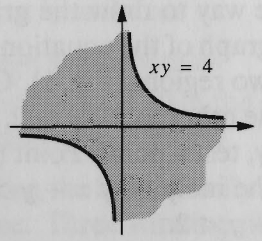
$x + y < 2$

Figure 3



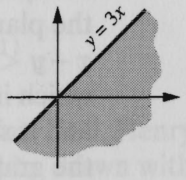
$|y - x| < 2$

Figure 4



$xy < 4$

Figure 5



$y < 3x$

Figure 6

SECTION 1-5 CONTINUOUS UNIFORM DISTRIBUTIONS

This is supposed to be a chapter on *discrete* probability. But there is a type of continuous problem that is so like the discrete case that I'm going to tell you about it now.

One-Dimensional Continuous Uniform Distributions

Trains stop at your station at 8:00 A.M., 8:15 A.M., and 8:30 A.M. You arrive at the station at random between 8:00 A.M. and 8:30 A.M. We'll find the probability that you have to wait only 5 minutes or less before a train arrives.

We choose as our mathematical model a sample space consisting of all points in the interval [0, 30] (or we could think in hours and use the interval [0, 1/2]). Figure 1 shows the total set of outcomes and the favorable arrival times (the times that lead to a wait of 5 minutes or less). All outcomes are equally likely (because you arrive at random) but we can't use

$$\frac{\text{favorable } \textit{number} \text{ of outcomes}}{\text{total } \textit{number} \text{ of outcomes}}$$

because there are an uncountably infinite number of each (that's why the problem is continuous rather than discrete). But in the same spirit we'll use

$$P(\text{wait 5 minutes or less}) = \frac{\text{favorable } \textit{length}}{\text{total } \textit{length}} = \frac{5+5}{30} = \frac{1}{3}$$

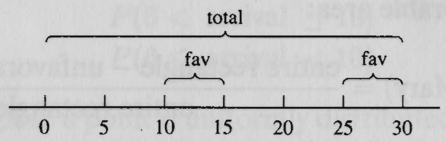


Figure 1

Suppose a point is chosen at random in an interval—the outcome of the experiment is said to be *uniformly distributed* on (or in) the interval. We choose as the mathematical model a sample space (universe) consisting of all points in the interval, where

$$P(\text{event}) = \frac{\text{favorable length}}{\text{total length}}$$

Two-Dimensional Continuous Uniform Distributions

John arrives at random between time 0 and time t_1, and Mary arrives at random between time 0 and t_2, where $t_1 < t_2$. Find the probability that John arrives before Mary.

Let x be John's arrival time and y be Mary's arrival time. Each outcome of the experiment is a *pair* of arrival times. The universe is the set of points in the rectangle in Fig. 2. The experiment amounts to picking a point at random from the rectangle.

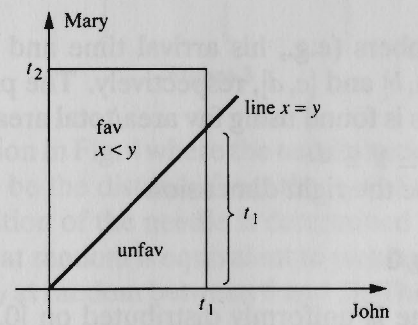


Figure 2

The favorable points are those in the rectangle where $x < y$ (see the review section if you don't remember how to graph inequalities).

For the 2-dimensional continuous analog of fav/total, we'll use

$$P(\text{John before Mary}) = \frac{\text{favorable area}}{\text{total area}}$$

Perhaps the easiest way to compute the fav area in Fig. 2 is indirectly, by first finding the unfavorable area:

$$P(\text{John before Mary}) = \frac{\text{entire rectangle} - \text{unfavorable triangle}}{\text{entire rectangle}}$$

$$= \frac{t_1 t_2 - \frac{1}{2}t_1^2}{t_1 t_2}$$

Suppose a point is chosen at random in a region in the plane—the outcome of the experiment is said to be *uniformly distributed* on (or in) the region.

As a special case, if a number x is chosen at random from the interval $[a, b]$ and a second number y is chosen (independently from the x choice) at random from the interval $[c, d]$, then the point (x, y) is uniformly distributed in a 2-dimensional rectangle.

We choose as the mathematical model a sample space (universe) consisting of all points in the region, where

$$P(\text{event}) = \frac{\text{favorable area}}{\text{total area}}$$

Warning

If *one* number is picked at random in an interval $[a, b]$, then probabilities involving that number are found using fav length/total length in the 1-dimensional universe $[a, b]$.

Suppose *two* numbers (e.g., his arrival time and her arrival time) are picked at random in $[a, b]$ and $[c, d]$, respectively. The probability of an event involving *both* numbers is found using fav area/total area in the 2-dimensional rectangle $a \le x \le b, c \le y \le d$.

Make sure you use the right dimension.

Events of Probability 0

Suppose an arrival time is uniformly distributed on $[0, 30]$. The probability of arriving *between* times 6 and 10 is $4/30$. But the probability of arriving *at* the one particular time 7 (or 3/4 or 8 or π) is 0, even though the event is *not* impossible. (This is different from most discrete situations.) Only events that have "length" have positive probabilities. Furthermore, when you ask for

the probability of arriving between 6 and 10, it doesn't matter whether you include the 6 and 10 or not (because the individual endpoints carry no probability). The following probabilities are all the same, namely, $4/30$:

$$P(6 \le \text{arrival} \le 10)$$
$$P(6 < \text{arrival} < 10)$$
$$P(6 < \text{arrival} \le 10)$$
$$P(6 \le \text{arrival} < 10)$$

Similarly, suppose a point is uniformly distributed in a region in 2-space. Then the probability that it lies in a given subregion is positive, namely, subregion area/total area. But the probability that the point specifically is $(3, 7)$ or (π, e) or $(\frac{2}{3}, 0)$ is 0, even though these events are not impossible. Only events that have "area" have positive probabilities.

Example 1 (Buffon's needle problem—a marvelous problem)

A needle is tossed onto a planked floor. The needle has length L, and the width of each plank is D where $L < D$. I'll find the probability that the needle hits a crack.

The needle can land angled in any one of four ways (Fig. 3). If I find that 40% of the needles in one particular orientation hit the crack, then by symmetry, 40% of the needles in any other orientation also hit the crack, and the probability in general that a needle hits the crack would be .4. So all I have to do is work with one particular orientation.

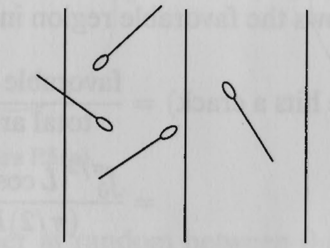


Figure 3

Let's use the orientation in Fig. 4 where the needle makes an acute angle θ with the horizontal. Let y be the distance from the eye of the needle to the right-hand crack. The position of the needle is determined by θ and y. Tossing the needle onto the floor at random is equivalent to picking θ at random between 0 and $\pi/2$ and picking y at random between 0 and D. The universe is a rectangle in θ, y space.

Figure 5 shows that a needle hits a crack if

$$L\cos\theta \ge y$$

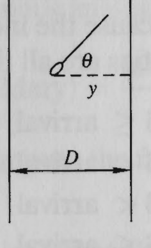


Figure 4

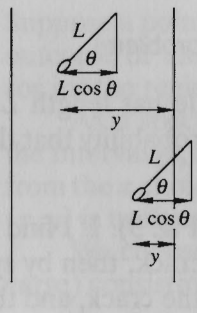


Figure 5

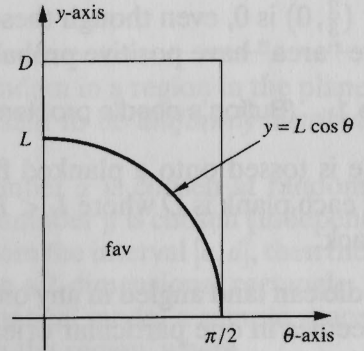


Figure 6

So the favorable region is the graph of $y \le L\cos\theta$, the region under the graph of $y = L\cos\theta$. Figure 6 shows the favorable region inside the universe. Then

$$P(\text{needle hits a crack}) = \frac{\text{favorable area}}{\text{total area}}$$

$$= \frac{\int_0^{\pi/2} L\cos\theta\, d\theta}{(\pi/2)D}$$

$$= \frac{L}{(\pi/2)D} = \frac{2L}{\pi D}$$

(As expected, the probability goes down if L gets smaller or if D gets larger.)

Problems for Section 1-5

1. A number x is chosen at random between -1 and 1 so that x is uniformly distributed in $[-1, 1]$. Find the prob that

(a) $-\frac{1}{2} < x < 0$ (c) $|x - .5| \le .1$

(b) $-\frac{1}{2} \le x \le 0$ (d) $3x^2 > x$

2. A point is chosen at random in a circle of radius 9. Find the prob that it's within distance 2 of the center.

3. Consider the quadratic equation $4x^2 + 4Qx + Q + 2 = 0$, where Q is uniformly distributed on $[0, 5]$. Find the prob that the roots are real.

4. Suppose θ is uniformly distributed on $[-\pi/2, \pi/2]$. Find the prob that $\sin\theta > \frac{1}{3}$.

5. Consider a circle with radius R. Here are two ways to choose a chord in the circle. In each case find the prob that the chord is longer than the side of an inscribed equilateral triangle

   (a) One end of the chord is point Q. The other end is determined by rotating the needle in the diagram by θ degrees where θ is uniformly distributed on $[0, 180]$.

   (b) Pick a point at random on the radius AB in the diagram (so that the indicated distance d is uniformly distributed on $[0, R]$). Draw a chord through the point perpendicular to AB.

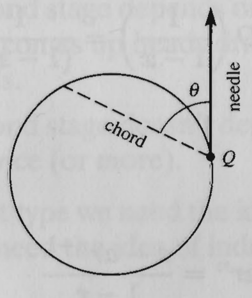


Figure P.5(a)

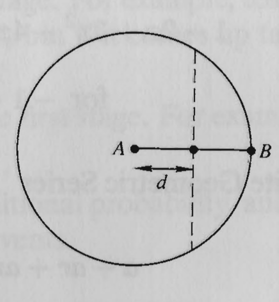


Figure P.5(b)

6. Choose a number at random between 0 and 1 and choose a second number at random between 1 and 3. Find the prob that

   (a) their sum is ≤ 3 (b) their product is > 1

7. John and Mary agree to meet and each arrives at random between 10:00 and 11:00. Find the prob that

   (a) the first to arrive has to wait more than 10 minutes for the other

   (b) Mary arrives at least 20 minutes before John

   (c) they arrive at the same time

REVIEW FOR THE NEXT CHAPTER

Series for e^x

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x \text{ for all } x$$

Series for e^{-x} (replace x by $-x$ in the e^x series)

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots = e^{-x} \text{ for all } x$$

Geometric Series

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad \text{for } -1 < x < 1$$

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad \text{for } -1 < r < 1$$

Differentiated Geometric Series

$$1 + 2x + 3x^2 + 4x^3 + \cdots = D\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

for $-1 < x < 1$

Finite Geometric Series

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r}$$

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}$$

Differentiated Finite Geometric Series

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = D\left(\frac{1 - x^{n+1}}{1-x}\right)$$

$$= \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}$$

CHAPTER 2

Independent Trials and 2-Stage Experiments

SECTION 2-1 CONDITIONAL PROBABILITY AND INDEPENDENT EVENTS

The aim of the chapter is to look at experiments with 2 (or more) stages. There are two types:

1. The second stage depends on the first stage. For example, toss a coin, and if it comes up heads draw one card, but if it comes up tails draw two cards.
2. The second stage doesn't depend on the first stage. For example, toss a coin twice (or more).

For the first type we need the idea of conditional probability, and for the second type we need the idea of independent events.

Conditional Probability

Draw a card from a deck. The probability that the card is a king is 4/52. But suppose you get a glimpse of the card, enough to tell that it's a picture. The probability that the card is a king *given that it is a picture* is denoted by $P(\text{king}|\text{picture})$ and is called a *conditional* probability. To find it, switch to a new universe consisting of the 16 pictures:

$$P(\text{king}|\text{picture}) = P(\text{king in the new universe}) = \frac{4 \text{ kings}}{16 \text{ pictures}} = \frac{1}{4}$$

Here's the informal definition of conditional probability.

(1) $P(B|A)$ is the probability that B occurs given that A has occurred. To find $P(B|A)$, cut the universe down to A, and then find the probability of B within the new universe.

For example, if 2 cards are drawn from a deck (without replacement) then

$$P(\text{spade on 2nd draw}|\text{spade on 1st draw}) = \frac{\text{12 spades left in deck}}{\text{51 cards left in deck}}$$

and

$$P(\text{spade on 2nd draw}|\text{heart on 1st draw}) = \frac{\text{13 spades left}}{\text{51 cards left}}$$

Formal Definition of Conditional Probability

If the condition A is more complicated than heart on 1st draw, it might not be so easy to work directly within the new universe. So we'll get a formula for conditional probabilities.

Here's an example to show the rationale behind the formula. Consider the universe in Fig. 1 where

$$P(A) = .6 \quad \text{and} \quad P(B) = .3$$

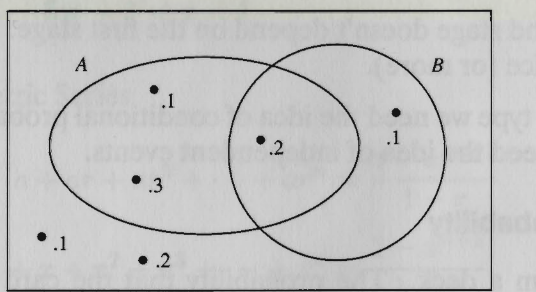


Figure 1

To assign a value to $P(B|A)$, consider A as the new universe. Within that new universe we see just one B-outcome, with probability .2. But a probability of .2 (w.r.t. the old universe where the total probability is 1) counts for more in the new universe, which only has a total probability of .6. It seems natural to choose

$$P(B|A) = \frac{.2}{.6}$$

Here's the idea in general:

(2) $$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{favorable prob within the } A \text{ world}}{\text{total prob of the } A \text{ world}}$$

Most books call (2) the formal definition of conditional probability.

If you can easily picture the new universe determined by the condition A, then use (1) to find P(B|A) directly. Otherwise, use (2), a safe, mechanical method.

The new A universe together with conditional probabilities assigned via (1) and/or (2) is a legitimate new probability space with all the usual properties; for example,

$$P(B|A) = 1 - P(\bar{B}|A)$$
$$P(\text{at least one of them}|A) = 1 - P(\text{none of them}|A)$$
$$P(B \text{ or } C|A) = P(B|A) + P(C|A) - P(B \text{ and } C|A)$$

etc.

Example 1

Find the prob that a poker hand contains 2 jacks if you already know it contains 1 ace (exactly 1 ace).

By (2),

$$P(\text{2J}|\text{1A}) = \frac{P(\text{2J and 1A})}{P(\text{1A})} = \frac{\binom{4}{2}\,4\binom{44}{2}/\binom{52}{5}}{4\binom{48}{4}/\binom{52}{5}} = \frac{\binom{4}{2}\binom{44}{2}}{\binom{48}{4}}$$

Example 2

Draw 2 cards without replacement. Find the prob of 2 jacks given that at least one of the cards is a picture.

$$P(\text{2J}|\text{at least one picture}) = \frac{P(\text{2J and at least one picture})}{P(\text{at least one picture})}$$

The event 2J and at least one picture is the same as the event 2J since a hand with 2J automatically has at least one picture. So

$$P(\text{2J}|\text{at least one picture}) = \frac{P(\text{2J})}{1 - P(\text{no pictures})} = \frac{\binom{4}{2}/\binom{52}{2}}{1 - \binom{36}{2}/\binom{52}{2}}$$
$$= \frac{\binom{4}{2}}{\binom{52}{2} - \binom{36}{2}}$$

Chain Rules for AND

The definition in (2) can be rewritten as

(3) $$\boxed{P(A \text{ and } B) = P(A)\,P(B|A)}$$

and in this form it can be thought of as a rule for AND.

The AND rule says that if A occurs, say, 20% of the time and B occurs 30% *of those times*, then A and B occur simultaneously 6% of the time.

By symmetry, we also have

(3′) $$\boxed{P(A \text{ and } B) = P(B)\,P(A|B)}$$

Similarly,

$$\boxed{P(A \,\&\, B \,\&\, C) = P(A)\,P(B|A)\,P(C|A \,\&\, B)}$$

and so on for as many events as you like.

(The event "A and B" is often denoted by AB.)

Example 3

Inside each box of Whizzo cereal is a card with 8 paint-covered circles. Underneath the paint, 3 of the circles contain the words "you," "win," and "prize," respectively. Figure 2 shows a typical card. If you can return the card to the Whizzo company with the paint scraped off precisely those three circles, you'll get a refund. (If you scrape the paint off a blank circle and try to paint over it again to cover your mistake, they'll be able to tell you cheated.) To what percentage of customers should the company be prepared to send refunds?

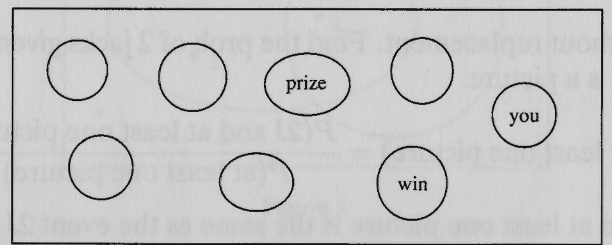


Figure 2

The experiment amounts to drawing 3 balls without replacement from a box containing 3 good balls and 5 bad balls. We want the probability of getting 3 goods.

Method 1

$$P(\text{3 good}) = \frac{\text{favorable committees}}{\text{total committees}} = \frac{1}{\binom{8}{3}} = \frac{3!\,5!}{8!} = \frac{3!}{8\cdot 7\cdot 6} = \frac{1}{56}$$

Be prepared for about 2% of the customers to win refunds.

Method 2 Use the chain rule for AND.

$$\begin{aligned} P(\text{3 good}) &= P(\text{good on 1st draw \& good on 2nd draw \& good on 3rd draw}) \\ &= P(\text{G on 1st})\,P(\text{G on 2nd}|\text{G on 1st})\,P(\text{G on 3rd}|\text{G on 1st \& 2nd}) \\ &= \frac{3}{8}\cdot\frac{2}{7}\cdot\frac{1}{6} \end{aligned}$$

Independent Events

If two events are unrelated so that the occurrence (or non-occurrence) of one of the events doesn't affect the likelihood of the other event, the events are called independent. We want to express this idea mathematically.

Here's the definition that's commonly given for independent events:

(4) $$\boxed{A \text{ and } B \text{ are independent iff } P(A \text{ and } B) = P(A)P(B).}$$

To see why (4) captures the intuitive idea of independence, combine it with (3) to get

(5) $$\boxed{\begin{aligned} A \text{ and } B \text{ are independent} \quad &\text{iff } P(B|A) = P(B) \\ &\text{iff } A \text{ has no effect on } B; \end{aligned}}$$

and similarly combine (4) with (3′) to get

(6) $$\boxed{\begin{aligned} A \text{ and } B \text{ are independent} \quad &\text{iff } P(A|B) = P(A) \\ &\text{iff } B \text{ has no effect on } A. \end{aligned}}$$

For example, suppose you toss a coin and draw a card. The sample space consists of $2 \cdot 52 = 104$ points such as heads and king of hearts, heads and queen of spades, and so on. Since the coin toss and card draw are intended to be independent, use (4) to assign probabilities to events:

$$P(\text{head and king of hearts}) = P(\text{head})P(\text{king of hearts}) = \frac{1}{2}\cdot\frac{1}{52} = \frac{1}{104}$$

$$P(\text{tail and heart}) = P(\text{tail})P(\text{heart}) = \frac{1}{2}\cdot\frac{13}{52} = \frac{1}{8}$$

The *three* events A, B, C are called independent *if all the following hold*:

$$\text{(7)}\qquad \begin{aligned} &P(A \text{ and } B \text{ and } C) = P(A)P(B)P(C) \\ &P(A \text{ and } B) = P(A)P(B) \\ &P(A \text{ and } C) = P(A)P(C) \\ &P(B \text{ and } C) = P(B)P(C) \end{aligned}$$

Similarly the *four* events A, B, C, D are independent if *all* the following hold:

$$\text{(8)}\qquad \begin{aligned} &P(A \text{ and } B \text{ and } C \text{ and } D) = P(A)P(B)P(C)P(D) \\ &P(\text{any 3 at a time}) = \text{product of separate probs} \\ &P(\text{any 2 at a time}) = \text{product of separate probs} \end{aligned}$$

> If balls (or cards) are drawn *with* replacement, then any event associated with one drawing and any event associated with another drawing are physically independent; we refer to the drawings themselves as independent.
> Similarly, successive coin (or die) tosses are independent.
> In such cases, use (4) and its generalizations in (7) and (8), and so on, to assign probabilities.

Example 4

Toss a biased coin 3 times where $P(\text{H}) = 1/3$. Find the prob of HHT in that order.

The tosses are independent, so

$$P(\text{HHT}) = P(\text{H})P(\text{H})P(\text{T}) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$$

Example 5

John and Mary take turns tossing one die; John goes first. The winner is the first player to throw a 4. Find the prob that John wins and the prob that Mary wins.

Method 1 Use the notation $\bar{4}\bar{4}4$ to represent the event

J throws a non-4, then M throws a non-4, and then J throws a 4.

Then

$$\begin{aligned} P(\text{J wins}) &= P(4 \text{ or } \bar{4}\bar{4}4 \text{ or } \bar{4}\bar{4}\bar{4}\bar{4}\bar{4}4 \text{ or } \ldots) \\ &= P(4) + P(\bar{4}\bar{4}4) + P(\bar{4}\bar{4}\bar{4}\bar{4}\bar{4}4) + \cdots \\ &\qquad \text{(events are mutually exclusive)} \\ &= P(4) + [P(\bar{4})]^2 P(4) + [P(\bar{4})]^4 P(4) + \cdots \\ &\qquad \text{(tosses are independent)} \\ &= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \left(\frac{5}{6}\right)^6 \frac{1}{6} + \cdots \end{aligned}$$

This is a geometric series (see the review before Chapter 2) with $a = 1/6$, $r = (5/6)^2$, so

$$P(\text{J wins}) = \frac{1/6y}{1 - (5/6)^2} = \frac{6}{11}.$$

And

$$P(\text{no one wins}) = P(\bar{4} \text{ forever}) = \left(\frac{5}{6}\right)^{\infty} = 0$$

$$P(\text{Mary wins}) = 1 - P(\text{John wins}) = \frac{5}{11}$$

Method 2 (slick) Let p be the prob that John wins. Then

$$\text{(9)}\qquad \begin{aligned} p &= P(4) + P(\bar{4}\bar{4} \text{ and then J throws the first 4 in the rest of the game}) \\ &= P(4) + P(\bar{4}\bar{4})P(\text{J throws the first 4 in the rest of the game} \mid \bar{4}\bar{4}) \end{aligned}$$

But the situation as the rest of the game begins (after $\bar{4}\bar{4}$) is the same as the situation at the beginning of the game where the prob that John wins is p. So

$$P(\text{J throws the first 4 in the rest of the game} \mid \bar{4}\bar{4}) = p$$

and (9) becomes

$$p = P(4) + P(\bar{4}\bar{4})p$$

$$p = \frac{1}{6} + \left(\frac{5}{6}\right)^2 p$$

Solve the equation to get the answer $p = 6/11$.

Probability of *A* Before *B*

Draw from a deck with or without replacement. Let's find the probability that a deuce is drawn before a picture.

Here's an intuitive argument. (Intuition can't always be trusted but in this case a more formal proof could be given to back it up.)

For all practical purposes, you must eventually draw a deuce or picture, that is, there *will* be a winning round: If the drawing is without replacement, this round will surely occur; if the drawing is with replacement, this round is not sure but occurs with probability 1 since the opposite event, 3–10 forever, has probability $\left(\frac{32}{52}\right)^{\infty} = 0$.[1] The most the other cards can do is delay the inevitable winning round; they have no effect on *who* wins, deuce or picture. So you might as well assume that the deck contains only the 4 deuces and 16 pictures to begin with, in which case the probability that you draw a deuce first is 4/20. In other words,

$$\begin{aligned}
&P(\text{deuce before picture as you draw repeatedly from the original deck})\\
&\quad = P(\text{D before P on the round a winner is decided})\\
&\quad = P(\text{D before P}|\text{no D's or P's have been drawn yet but one is drawn } now)\\
&\quad = P(\text{D on one draw from a new 20 card universe of D's and P's})\\
&\quad = \frac{4}{20}
\end{aligned}$$

Here's a general rule for $P(A \text{ before } B)$ where A and B are two of the possible (disjoint) outcomes of an experiment and the experiment is performed over and over independently.

Drawings *with* replacement are one instance of repeated independent experiments so they are covered by the rule. Drawings with*out* replacement are not in this category, but it just so happens that the rule holds in this case too (as the deuce/picture example showed).

> $P(A \text{ occurs before } B)$
> = prob of A in *one* trial in a new universe where *only* A and B can occur
> Equivalently,
> $$P(A \text{ occurs before } B) = \frac{P(A)}{P(A)+P(B)} = \frac{\text{fav prob in an } A, B \text{ only world}}{\text{total prob in the } A, B \text{ world}}$$

Problems for Section 2-1

1. Draw 2 cards from a deck without replacement. Find
   (a) P (second is a queen|first is a queen)
   (b) P(second is a queen|first is an ace)
   (c) P(first is higher or the 2 cards tie|first is a king) (Remember that aces are high.)
   (d) P(1 ace|first is an ace)

[1] The event 3–10 forever is *possible* because it *does* contain points, for example, 535353..., 34444..., 10^{∞} and so on. But it has probability 0.

2. Given P(rain|Jan. 7) = .3, find whichever of the following are possible.
   (a) P(not rain|Jan. 7) (b) P(rain|not Jan. 7)
3. Toss 2 dice. Find the prob that the first is 6 given that the sum is 8.
4. Find the prob that a poker hand contains at least 1 king given that it contains at least 1 ace.
5. The North and South partners in bridge have 9 spades between them. Find the prob that the 4 spades held by the East-West pair split 3–1 (East has 3, West has 1, or vice versa).
6. A point is chosen at random from a unit square $ABCD$. Find the prob that it's in triangle ABD given that it's in triangle ABC.
7. A box contains 10 white, 9 black, and 5 red balls. Draw 4 without replacement.
   (a) Find the prob of BRBW (in that order).
   (b) Find P(BBRW).
   (c) Repeat part (a) if every time a red is drawn not only is it returned to the box but another red is added as well.
   (d) Find the prob of 2B, 1W, 1R.
   (e) Find the prob of W on the 4th draw.
   (f) Find the prob of W on the last two draws.
8. You send out three messages for help:

   a smoke signal that has prob .1 of being seen
   a message in a bottle that has prob .2 of being found
   a carrier pigeon that has prob .3 of arriving

   Find the prob you are saved, assuming that smoke, bottles, and pigeons are independent.
9. Switches I and II in the diagram work independently. The prob is .7 that switch I is closed (allowing a signal to get through) and the prob is .2 that switch II is closed.

   If a signal from A to B doesn't arrive, find the prob that switch II was open.

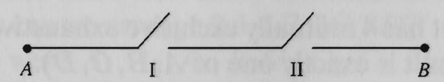


Figure P.9

10. (slippery) Two balls are painted independently, white with prob 1/2 and black with prob 1/2, and then placed in a box. Find the prob that both balls are black if
    (a) you see a wet black paintbrush lying around
    (b) you draw a ball from the box and it's black

11. The prob that a missile hits its target is .8. Missiles are fired (independently) at a target until it is hit. Find the prob that it takes more than 3 missiles to get the hit.

12. A box contains 10 white balls and 5 black balls. Draw 4. Find the prob of getting W on the 1st and 4th draws if the drawing is

(a) without replacement (b) with replacement

13. Players A, B, C toss coins simultaneously. The prob of heads is p_a for A, p_b for B, and p_c for C.

If the result is 2H and 1T or the result is 2T and 1H, then the player that is different from the other two is called the odd man out and the game is over. If the result is 3H or 3T, then the players toss again until they get an odd man out.

Find the prob that A will be the odd man out.

14. Draw cards. Find the prob of getting a heart before a black card.

15. Keep tossing a pair of dice. Find the prob of getting 5 before 7.

16. There are 25 cars in the parking lot, with license numbers $C_1, C_2, \ldots, C_{25}$. Assume the cars leave at random. Find the prob that

(a) C_1 leaves before C_2

(b) C_{12} leaves before C_2 and C_3

(c) C_1 leaves before C_2 which in turn leaves before C_3

17. A die is biased so that $P(1) = .2$ and $P(2) = .3$. Toss repeatedly. Find the prob of getting a 1 before a 2.

SECTION 2-2 THE BINOMIAL AND MULTINOMIAL DISTRIBUTIONS

Now we're ready to look at multi-stage experiments, beginning with the kind where the stages are independent. In particular we'll examine what happens when the same experiment is repeated over and over independently.

The Multinomial Distribution

Suppose an experiment has 4 mutually exclusive exhaustive outcomes A, B, C, D (at each trial the result is exactly one of A, B, C, D).

Repeat the experiment, say, 9 times so that we have 9 independent trials. We'll show that

$$(1) \quad P(3A, 1B, 3C, 2D) = \frac{9}{3!\,1!\,3!\,2!}\,[P(A)]^3\,[P(B)]^1\,[P(C)]^3\,[P(D)]^2$$

The general formula for n independent trials (instead of 9), where each trial has r possible outcomes (instead of 4), has the same pattern as (1). We say that the result of the n trials has a *multinomial distribution.*

Here's why the formula in (1) holds.

The sample space consists of 4^9 points, namely, all strings of length 9 using the letters A, B, C, D. The point $DDDDDDDDB$, for instance, corresponds to D on the first 8 trials followed by B on the 9th trial. Figure 1 shows some of the outcomes in the event "$3A, 1B, 3C, 2D$" (the favorable outcomes).

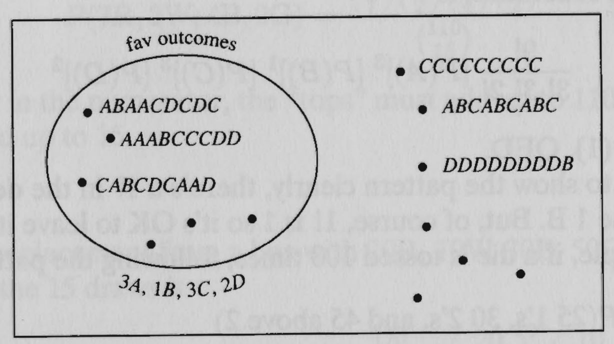


Figure 1 The universe

The 4^9 outcomes in the universe are not equally likely; for example, the outcome $CCCCCCCCC$ has probability $[P(C)]^9$, while the outcome $AAAAAAAAA$ has probability $[P(A)]^9$. So we can't use fav/total. Instead, we'll find the probability of each favorable outcome and add them all up.

One of the *favorable* outcomes is $ABAACDCDC$. The trials are independent, so

$$(2) \quad \begin{aligned} P(ABAACDCDC) &= P(A)P(B)P(A)P(A)P(C)P(D) \\ &\quad P(C)P(D)P(C) \\ &= [P(A)]^3\,[P(B)]^1\,[P(C)]^3\,[P(D)]^2 \end{aligned}$$

Similarly, *each* favorable outcome has the probability in (2).

Now we need to know how many favorable outcomes there are; that is, in how many ways can we arrange $3A, 1B, 3C, 2D$?

If the letters were $A_1, A_2, A_3, B, C_1, C_2, C_3, D_1$, and D_2 we would have 9! permutations. Having 3 identical A's instead of A_1, A_2, A_3 means that the 9! is too large by a factor of 3!. Having 3 identical C's instead of C_1, C_2, C_3 makes the 9! too large by another factor of 3!. And having 2 identical D's instead of D_1, D_2 makes the 9! too large by a factor of 2!. So there are 9!/(3! 3! 2!) permutations.

(Here's another way to find the number of permutations. To line up the 9 letters, pick 3 places in the lineup for the 3 identical A's, pick one place for the B, pick 3 places for the 3 identical C's, then the last 2 places must go to the D's. This can be done in

$$\binom{9}{3} \cdot 6 \cdot \binom{5}{3} = \frac{9!}{3!\,6!} \cdot 6 \cdot \frac{5!}{3!\,2!} = \frac{9!}{3!\,3!\,2!}$$

ways.

So the event $3A, 1B, 2C, 2D$ consists of $9!/(3!\,3!\,2!)$ outcomes each with probability $[P(A)]^3\,[P(B)]^1\,[P(C)]^3\,[P(D)]^2$. The probability of the event, the sum of these probabilities, is

$$\frac{9!}{3!\,3!\,2!}\,[P(A)]^3\,[P(B)]^1\,[P(C)]^3\,[P(D)]^2$$

the formula in (1), QED.

(In order to show the pattern clearly, there's a 1! in the denominator in (1) to match the 1 B. But, of course, 1! is 1 so it's OK to leave it out.)

For example, if a die is tossed 100 times, following the pattern in (1),

$$P(25\text{ 1's, 30 2's, and 45 above 2})$$
$$= \frac{100!}{25!\,30!\,45!}\,[P(1)]^{25}\,[P(2)]^{30}\,[P(\text{above }2)]^{45}$$
$$= \frac{100!}{25!\,30!\,45!}\left(\frac{1}{6}\right)^{25}\left(\frac{1}{6}\right)^{30}\left(\frac{4}{6}\right)^{45}$$
$$= \frac{100!}{25!\,30!\,45!}\left(\frac{1}{6}\right)^{55}\left(\frac{4}{6}\right)^{45}$$

Some Typical Independent Trials

1. Coin tosses, die tosses
2. Tossing balls into boxes
3. Drawings *with* replacement
4. Drawings with*out* replacement from a *large* population (as in polling)

Actually, drawings with*out* replacement are *not* independent, but if the population is large, then one draw has such a slight effect on the next draw that for all practical purposes, they are independent and you can use the multinomial distribution to find probs.

The Classical Urn Problem

Draw 15 balls from a box containing 20 red, 10 white, 30 black, and 50 green. I'll find the probability of 7R, 2W, 4B, 2G.

If the drawing is *with replacement*, then there are 15 independent trials, and on any one trial, $P(R) = 20/110$, $P(W) = 10/110$, $P(B) = 30/110$, $P(G) = 50/110$. So

$$P(7R, 2W, 4B, 2G) = \frac{15!}{7!\,2!\,4!\,2!}\left(\frac{20}{110}\right)^7\left(\frac{10}{110}\right)^2\left(\frac{30}{110}\right)^4\left(\frac{50}{110}\right)^2$$

On the other hand, if the 15 balls are drawn *without replacement*, then they are a committee and

$$P(7R, 2W, 4B, 2G) = \frac{\binom{20}{7}\binom{10}{2}\binom{30}{4}\binom{50}{2}}{\binom{110}{15}}$$

Note that in the numerator, the "tops" must add up to 110 and the "bottoms" must add up to 15.

Example 1

Draw 15 with replacement from a box with 20R, 10W, 30B, 50G. The prob of 7R and 2W in the 15 draws is

$$P(7R, 2W) = P(7R, 2W, 6\text{ others}) = \frac{15!}{7!\,2!\,6!}\left(\frac{20}{110}\right)^7\left(\frac{10}{110}\right)^2\left(\frac{80}{110}\right)^6$$

In a multinomial problem, make sure you remember the others if there are any.

On the other hand, the prob of 7R, 8W in the 15 draws is

$$P(7R, 8W) = P(7R, 8W, \text{no others}) = \frac{15!}{7!\,8!\,0!}\left(\frac{20}{110}\right)^7\left(\frac{10}{110}\right)^8\left(\frac{80}{110}\right)^0$$
$$= \frac{15!}{7!\,8!}\left(\frac{20}{110}\right)^7\left(\frac{10}{110}\right)^8$$

In this case, where there are no others, you might as well ignore them right from the beginning. There's no point in sticking in 0! and $(80/110)^0$ only to have them disappear again anyway.

Example 2

At the University of Illinois, 30% of the students are from Chicago, 60% are from downstate, and 10% are out of state. Find the probability that, if 10 students are picked at random by the Daily News to give their indignant reaction to the latest tuition increase, 6 are Chicagoans and 4 are out-of-staters.

The students are sampled with*out* replacement from a *large* population, so they can be treated as independent trials. Using the multinomial distribution,

$$P(6\text{ Chicagos, 4 out-of-states}) = \frac{10!}{6!\,4!}(.3)^6(.1)^4$$

Bernoulli Trials and the Binomial Distribution

Suppose an experiment has two outcomes, titled success and failure (e.g., tossing a coin results in heads or tails, testing a light bulb results in accept or reject, tossing a die results in 4 versus non-4). Independent repetitions of the experiment are called *Bernoulli trials.*

Suppose that in any one trial

$$P(\text{success}) = p, P(\text{failure}) = 1 - p = q.$$

Then, as a special case of the multinomial distribution,

$$P(k \text{ successes in } n \text{ trials}) = P(k \text{ successes}, n-k \text{ failures}) = \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

The coefficient can be written as $\binom{n}{k}$ so that

$$(3) \qquad \boxed{P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k}}$$

To repeat, here's *why* (3) holds. Each outcome consisting of k successes and $n-k$ failures has probability $p^k q^{n-k}$. The coefficient $\binom{n}{k}$ counts how *many* of these outcomes there are (as many as there are ways of picking k out of the n spots for the successes).

We say that the number of successes in n Bernoulli trials, where $P(\text{success}) = p$, has a *binomial distribution* with parameters n and p, meaning that (3) holds.

Example 3

If a fair coin is tossed 10 times, then, by (3),

$$P(2\text{H}) = \binom{10}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^8 = \binom{10}{2}\left(\frac{1}{2}\right)^{10}$$

Example 4

A dealer sells 10 new cars. The prob that a new car breaks down is .3.

(a) Find the prob that 8 of the cars work and 2 break down.

(b) Find the prob that the first 2 cars sold break down and the others work.

The cars are Bernoulli trials. Each trial results in breaks or works.

(a) $P(8 \text{ work in } 10 \text{ trials}) = \binom{10}{8}(.7)^8(.3)^2$

(b) $P(\text{BBWWWWWWWW}) = (.3)^2(.7)^8$

Example 5

Find the prob that a 6-letter word contains two Z's.

The positions in the words are independent trials since repetition is allowed. Each trial has two possible outcomes: Z with prob 1/26 and other with prob 25/26. The number of Z's has a binomial distribution, so

$$P(2Z) = \binom{6}{2}\left(\frac{1}{26}\right)^2\left(\frac{25}{26}\right)^4$$

All or None Problems

An urn contains 20 white, 30 black, and 40 red balls. Draw 5 with replacement.

If you want the probability that all 5 are white, you can use the binomial distribution to get

$$P(\text{all white}) = P(5\text{W}) = \binom{5}{5}\left(\frac{20}{90}\right)^5\left(\frac{70}{90}\right)^0 = \left(\frac{20}{90}\right)^5$$

But it's faster to simply use

$$P(\text{all white}) = P(\text{WWWWW}) = \left(\frac{20}{90}\right)^5 \quad \text{(since the trials are ind)}$$

Similarly, the fastest way to get the probability of no whites is

$$P(\text{no whites}) = P(\bar{\text{W}}\bar{\text{W}}\bar{\text{W}}\bar{\text{W}}\bar{\text{W}}) = \left(\frac{70}{90}\right)^5$$

Example 6

A coin is biased so that $P(\text{H}) = .6$. If the coin is tossed 10 times,

$$P(\text{at least 2H}) = 1 - P(\text{no H}) - P(1\text{H}) = 1 - (.4)^{10} - \binom{10}{1}(.6)\,(.4)^9$$

The Geometric Distribution

Let $P(\text{heads}) = p$. Toss until a head turns up. Here's how to find the probability that it takes 10 tosses:

$$\begin{aligned} &P(\text{it takes 10 tosses to get a head}) \\ &= P(\text{first 9 tosses are tails and 10th is head}) \\ &= (1-p)^9 p \end{aligned}$$

The number of trials to get the first success in Bernoulli trials where $P(\text{success}) = p$ is said to have a *geometric* distribution with parameter p.

The Negative Binomial Distribution

Let $P(\text{heads}) = p$. Toss until there is a total of 17 heads. Here's how to find the probability that the game ends on the 30th toss:

$$\begin{aligned} &P(\text{it takes 30 tosses to get 17 heads}) \\ &= P(\text{16 heads in 29 tosses and heads on the 30th}) \\ &= P(\text{16 heads in 29 tosses})P(\text{H on 30th}) \quad \text{(since tosses are ind)} \\ &= \binom{29}{16} p^{16}(1-p)^{13} p \end{aligned}$$

The number of trials needed to get r successes in Bernoulli trials where $P(\text{success}) = p$ is said to have a *negative binomial distribution* with parameters r and p.

Problems for Section 2-2

1. Of all the toasters produced by a company, 60% are good, 30% are fair, 7% burn the toast, and 3% electrocute their owners. If a store has 40 of these toasters in stock, find the prob that they have

(a) 30 good, 5 fair, 3 burners, 2 killers **(c)** no killers
(b) 30 good, 4 fair

2. Toss 16 nickels. Find the prob of

(a) no heads **(b)** 7 heads **(c)** at least 15 heads

3. Five boxes each contain 7 red and 3 green balls. Draw 1 ball from each box. Find the prob of getting more green than red.

4. Sixty percent of the country is Against, 30% is For, and 10% is Undecided. If 5 people are polled find the prob that

(a) all are For
(b) 1 is For, 2 are Against, and 2 are Undecided
(c) a majority is Against

5. A couple has 6 children. Find the prob that they have

(a) 3 girls and 3 boys **(c)** GBGGBB in that order
(b) 3 girls first and then 3 boys

6. A coin is biased so that $P(\text{H}) = .6$. Toss it 10 times.

(a) Find the prob of 6 heads overall given that the second toss is tails.
(b) Find the prob of at least 9 heads given that you got at least 8 heads.

7. Toss 10 balls at random into 5 boxes. Find the prob that

(a) each box gets 2 balls **(c)** box B_3 gets 6 balls
(b) box B_2 is empty **(d)** no box is empty

8. A machine has 7 identical independent components. The prob that a component fails is .2. In order for the machine to operate, 5 of its 7 components must work. Find the prob that the machine fails.

9. Find the prob that a 7-digit string contains

(a) two 4's **(b)** exactly one digit > 5

10. (poker dice) Toss 5 dice. Find the prob of getting a pair (and nothing better than a pair).

11. Ten pieces of candy are given out at random in a group of 5 boys, 7 girls, and 9 adults. Find the prob that 4 pieces of candy go to the girls if

(a) people are allowed to get more than one piece
(b) no one is allowed to get more than one piece

12. If 40% of marriages end in divorce and we assume that divorces are independent of one another, find the prob that of 8 couples

(a) only the Smiths and Joneses will stay married
(b) exactly 2 of the 8 couples will stay married

13. At a particular intersection with a stop sign you observe that 1 out of every 20 cars fails to stop. Find the prob that among the next 100 cars at least 3 don't stop.

14. Toss 6 balls at random into 10 boxes. Find the prob that

(a) they split 4–2 (4 go into one box and 2 into a second box)
(b) they split 3–3
(c) they all go into different boxes

15. A drawer contains 10 left gloves and 12 right gloves. If you pull out a handful of 4 gloves, what's the prob of getting 2 pairs (2L and 2R)?

16. Fifteen percent of the population is left-handed. If you stop people on the street what's the prob that

(a) it takes at least 20 tries to get a lefty
(b) it takes exactly 20 tries to get a lefty
(c) it takes exactly 20 tries to get 3 lefties
(d) it takes at least 20 tries to get 3 lefties (the more compact your answer, the better)
(e) the number of tries to get a lefty is a multiple of 5

17. A coin has $P(\text{head}) = p$. Find the prob that it takes 10 tosses to get a head and a tail (i.e., at least one of each).

SECTION 2-3 SIMULATING AN EXPERIMENT

A box contains 12 red balls and 8 black balls. Draw 10 times without replacement. Then

$$P(6 \text{ reds}) = \frac{\binom{12}{6}\binom{8}{4}}{\binom{20}{10}} \approx .35$$

The physical interpretation of the mathematical model is that if you do this 10-draw experiment many times, it is likely (but not guaranteed) that the percentage of times you'll get 6 reds will be close to (but not necessarily equal to) 35%.

(Notice how much hedging there has to be in the last paragraph. *Within* a mathematical model, theorems can be stated precisely and proved to hold. But when you try to apply your model to the real world, you are stuck with imprecise words like *many*, *very likely*, and *close to*.)

The random number generator in a computer can be used to simulate drawing balls from a box so that you can actually do the 10-draw experiment many times. The program that follows was done with Mathematica.

The subprogram enclosed in the box is a single 10-draw experiment: It draws 10 balls without replacement from an urn containing 12 reds and 8 blacks. When it's over, the counter named Red tells you the total number of reds in the sample. Here's how this part works.

- Start the Red counter at 0.
- Draw the first ball by picking an integer z at random between 1 and 20.
- The integers from 1 to 12 are the red balls; the integers from 13 to 20 are the blacks.
- Step up Red by 1 if z is a red ball, that is, if $z \leq 12$ (in Mathematica this has to be written as $z <= 12$).
- On the second draw pick an integer z at random between 1 and 19.
- If the first draw was *red* then the integers from 1 to 11 are the red balls, the integers from 12 to 19 are the blacks.
- If the first draw was *black* then the integers from 1 to 12 are the red balls, the integers from 13 to 19 are the blacks.
- Again, the counter Red is stepped up by 1 if z is red.
- And so on through the 10 draws.

The program as a whole repeats the 10-draw experiment n times. The counter named SixReds keeps track of how often you get 6 reds in the 10 draws. The final output divides SixReds by n to get the fraction of the time that this happens, that is, the relative frequency of 6 red results.

Here's what happened when I entered the program and ran it twice.

```
In[1]

Percent6RedWO[n_] :=

(SixReds = 0;
```



```
Do[

   (Red = 0;
    total = 20;
    Do [z = Random[Integer,{1,total}];
        If [z <= 12 - Red, Red = Red + 1];
        total = total - 1,
       {10}
       ]
   );

  If[Red == 6, SixReds = SixReds + 1],

  {n}

 ];

SixReds/n//N    (*Here is the output;

                  The N makes it a decimal rather than a fraction*)

)

In[2]

Percent6RedWO[500] (*Repeat the ten-draw experiment 500 times*)

Out[2]

0.308
```

For that run, 31% of the time I got 6 reds in 10 draws.

```
In[3]

Percent6RedWO[1000]  (*Repeat the ten-draw experiment 1000 times*)

Out[3]

0.335
```

For the second run, 34% of the time I got 6 reds in 10 draws.

If the drawing is done *with* replacement, then

$$P(6 \text{ reds in } 10 \text{ draws}) = \binom{10}{3}\left(\frac{12}{20}\right)^6\left(\frac{8}{20}\right)^4 \approx .25 \text{ (binomial distribution)}$$

Here's the program adjusted so that it simulates n 10-draw experiments where the drawing is *with* replacement. Only the boxed subprogram is changed: now it just picks 10 integers at random between 1 and 20 where on each draw the integers 1 to 12 are red.

```
In[4]

Percent6RedWITH[n_] :=

(SixReds = 0;
 Do[
   (Red = 0;
    total = 20;
    Do [z = Random[Integer,{1,total}];
      If [z <= 12, Red = Red + 1];
      {10}
      ]
   );

   If[Red == 6, SixReds = SixReds + 1],
   {n}
   ];
SixReds/n//N
)

In[5]

Percent6RedWITH[1000]

Out[5]

0.265
```

SECTION 2-4 THE THEOREM OF TOTAL PROBABILITY AND BAYES' THEOREM

This section is about 2-stage (or multi-stage) experiments where the second stage depends on the first.

The Theorem of Total Probability

Here's a typical 2-stage experiment.

(1) A box contains 2 green and 3 white balls. Draw 1.
If the ball is green, draw a card from a fair deck.
If the ball is white, draw a card from a deck consisting of just the 16 pictures.

We'll find the probability of drawing a king.

The tree diagram in Fig. 1 describes the situation. The labels on the first set of branches are

$$P(\text{green}) = \frac{2}{5} \quad \text{and} \quad P(\text{white}) = \frac{3}{5}$$

The labels on the second set of branches are

$$P(\text{king}|\text{green}) = \frac{4}{52}, \qquad P(\text{non-king}|\text{green}) = \frac{48}{52}$$

$$P(\text{king}|\text{white}) = \frac{4}{16}, \qquad P(\text{non-king}|\text{white}) = \frac{12}{16}$$

Note that at each vertex of the tree, the sum of the probabilities is 1.

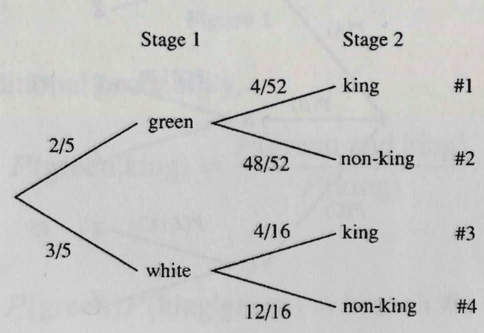


Figure 1

Since one of green and white has to occur at the first stage,

$$P(\text{king}) = P(\text{green and king or white and king})$$
$$= P(\text{green and king}) + P(\text{white and king})$$
(by the OR rule for mutually exclusive events)
$$= P(\text{green})P(\text{king}|\text{green}) + P(\text{white})P(\text{king}|\text{white})$$
(by the AND rule)
$$= \frac{2}{5}\cdot\frac{4}{52} + \frac{3}{5}\cdot\frac{4}{16}$$

If we use the notation

$$\text{branch \#1} = \text{product of the probabilities along the branch} = \frac{2}{5} \cdot \frac{4}{52}$$

then the answer can be written as

$$P(\text{king}) = \#1 + \#3 = \text{sum of favorable branches}$$

Here's the general rule, called the *theorem of total probability*.

If at the first stage the result is exactly one of A, B, C, then the probability of Z at the second stage is

(2) $P(Z) = P(A)P(Z|A) + P(B)P(Z|B) + P(C)P(Z|C)$

And here's a restatement that makes it easy to use.

(3) $P(Z)$ = sum of favorable branches in Fig. 2 = #1 + #3 + #5

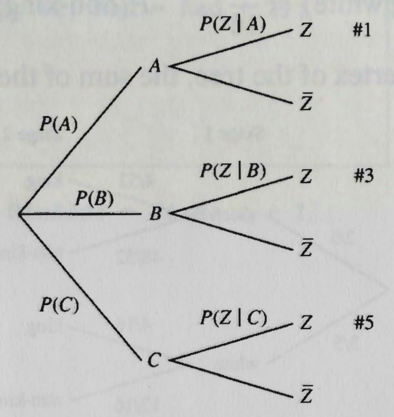


Figure 2

The theorem of total probability as stated in (2) can be thought of in a more general context, without reference to a 2-stage experiment: If a probability space can be divided into, say, 3 mutually exclusive exhaustive events A, B, C, then (2) holds for any event Z.

Bayes' Theorem

Let's use the same experiment again:

A box contains 2 green and 3 white balls. Draw 1 ball.
If the ball is green, draw a card from a fair deck.
If the ball is white, draw a card from a deck consisting of just the 16 pictures.

Suppose you draw a king on the second stage (Fig. 1 again). We'll go backward and find the probability that it was a green ball on the first stage. In other words, we'll find

$$P(\text{green on first stage}|\text{king on second})$$

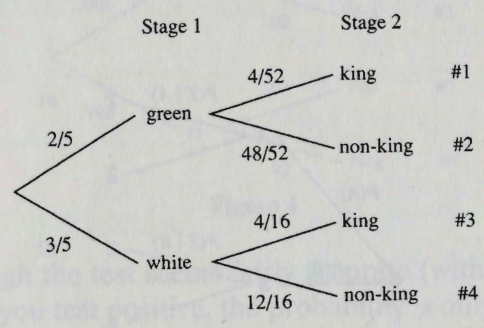


Figure 1

By the rule for conditional probability,

$$P(\text{green}|\text{king}) = \frac{P(\text{green and king})}{P(\text{king})}$$

The numerator is

$$P(\text{green})P(\text{king}|\text{green}) = \text{branch \#1}$$

By the theorem of total probability, the denominator is #1 + #3. So

$$P(\text{green}|\text{king}) = \frac{\#1}{\#1 + \#3} = \frac{\text{fav king branches}}{\text{total king branches}}$$

$$= \frac{\frac{2}{5} \cdot \frac{1}{13}}{\frac{2}{5} \cdot \frac{1}{13} + \frac{3}{5} \cdot \frac{1}{4}} = \frac{8}{47}$$

Here's the general rule, called *Bayes' Theorem.*

The a posteriori (backward conditional) probability of A at the *first* stage, given Z on the *second* stage (Fig. 3), is

$$P(A|Z) = \frac{P(A \text{ and } Z)}{P(Z)}$$

$$(4) \qquad = \frac{Z\text{-branches that are favorable to } A}{\text{total } Z\text{-branches}}$$

$$= \frac{\#1}{\#1 + \#3 + \#5}$$

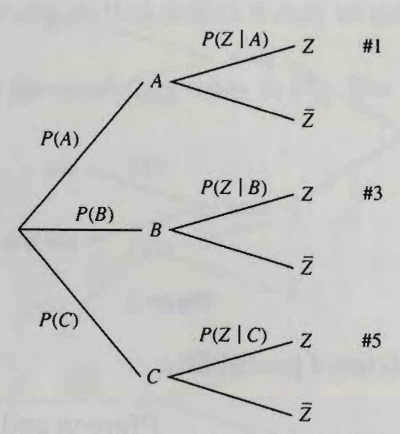


Figure 3

Example 1

Suppose $\frac{1}{2}\%$ of the population has a disease D. There is a test to detect the disease. A positive test result is supposed to mean that you have the disease, but the test is not perfect. For people *with* D, the test misses the diagnosis 2% of the time; that is, it reports a false negative. And for people with*out* D, the test incorrectly tells 3% of them that they have D; that is, it reports a false positive.

(a) Find the probability that a person picked at random will test positive.

(b) Suppose your test comes back positive. What is the (conditional) probability that you have D?

(a) Figure 4 shows the tree diagram. By the theorem of total probability,

$$P(\text{positive}) = \#1 + \#3 = (.005)(.98) + (.995)(.03)$$

(b) By Bayes' theorem,

$$P(\text{D}|\text{pos}) = \frac{\text{favorable pos branches}}{\text{total pos branches}} = \frac{\#1}{\#1 + \#3}$$

$$= \frac{(.005)(.98)}{(.005)(.98) + (.995)(.03)} \sim .14$$

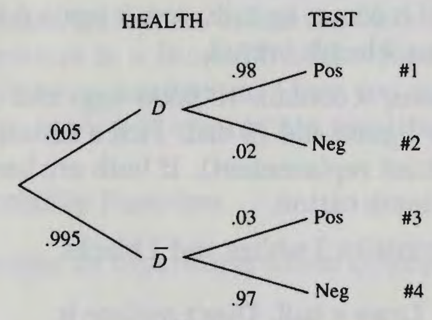


Figure 4

So even though the test seems fairly accurate (with success rates of 98% and 97%), if you test positive, the probability is only .14 that you actually have the disease. (The probability came out low because so few people have the disease to begin with.)

What we would really like to know in this situation is a *first stage* result: Do you have the disease? But we can't get this information without an autopsy. *The first stage is hidden*. But the second stage (the result of the test) is not hidden. The best we can do is make a prediction about the first stage by looking at the second stage. This illustrates why backward conditionals are so useful.

Problems for Section 2-4

1. The prob of color blindness is .02 for a man and .001 for a woman. Find the prob that a person picked at random is color blind if the population is 53% men.

2. Draw a card. If it's a spade, put it back in the deck and draw a second card. If the first card isn't a spade, draw a second card without replacing the first one. Find the prob that the second card is the ace of spades.

3. A multiple-choice exam gives 5 choices per question. On 75% of the questions, you think you know the answer; on the other 25% of the questions, you just guess at random. Unfortunately when you *think* you know the answer, you are right only 80% of the time (you dummy).

(a) Find the prob of getting an arbitrary question right.
(b) If you do get a question right, what's the prob that it was a lucky guess?

4. Box A has 10 whites and 20 reds, box B has 7 whites and 8 reds, and box C has 4 whites and 5 reds. Pick a box at random and draw one ball. If the ball is white, what's the prob that it was from box B?

5. Toss a biased coin where $P(\text{H}) = 2/3$. If it comes up heads, toss it again 5 times. If it comes up tails, toss it again 6 times. Find the prob of getting at least 4 heads overall.

6. Of 10 egg cartons, 9 contain 10 good eggs and 2 bad while a tenth carton contains 2 good and 10 bad. Pick a carton at random and pull out 2 eggs (without replacement). If both are bad, find the prob that you picked the tenth carton.

7. A box of balls contains 3 whites and 2 blacks.

round 1	Draw a ball. Don't replace it.
round 2	If the ball is white, toss a fair coin.
	If the ball is black, toss a biased coin where $P(\text{H}) = .8$.
round 3	If heads, draw 2 balls from the (depleted) box.
	If tails, draw 1 ball.

Find the prob of getting at least 1 white at round 3.

8. An insurance company unofficially believes that 30% of drivers are careless and the prob that a driver will have an accident in any one year is .4 for a careless driver and .2 for a careful driver. Find the prob that a driver will have an accident next year given that she has had an accident this year.

9. Look at the accompanying tree diagram. What is each of the following the probability of?

(a) .2 (b) .3 (c) (.2)(.3)

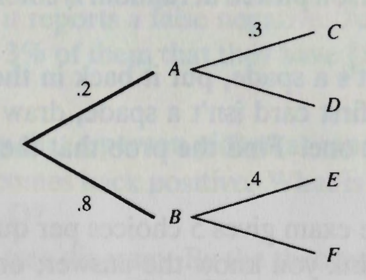


Figure P.9

10. Toss a die 3 times. Find the probability that the result of the third toss is larger than each of the first two.

Suggestion: Condition on the third toss and use the theorem of total probability.

SECTION 2-5 THE POISSON DISTRIBUTION

In Section 2.2 we found a formula, called the binomial distribution, for the probability of k successes in n Bernoulli trials. Now we'll look at a related formula called the Poisson distribution. Once you see how it's connected to the binomial, you can apply it in appropriate situations.

The Poisson Probability Function

Let λ be fixed. Consider an experiment whose outcome can be $0, 1, 2, 3, \ldots$. If

$$P(\text{outcome is } k) = \frac{e^{-\lambda}\lambda^k}{k!} \text{ for } k = 0, 1, 2, 3, \ldots \tag{1}$$

then we say that the outcome has a Poisson distribution with parameter λ. The formula in (1) is the *Poisson probability function.*

The Poisson prob function is legitimate because the sum of the probs is 1, as it should be:

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \underbrace{\sum_{n=0}^{\infty} \frac{\lambda^k}{k!}}_{e^{\lambda}} = 1$$

Here we used the standard series from calculus

$$e^x = \sum_{n=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The Poisson Approximation to the Binomial

Consider n Bernoulli trials where $P(\text{success}) = p$.

Let $\lambda = np$, interpreted as the average number of successes to be expected in the n trials.

(To see the physical interpretation of λ, consider tossing a coin 200 times where $P(\text{heads}) = .03$, so that $n = 200$, $p = .03$. If many people toss 200 times each, one person might get no heads and another might get 199 heads, but it is very likely that the *average* number of heads per person is near $.03 \times 200 = 6$.)

Suppose n is large, p is small, their product λ is moderate, and k is much smaller than n. We'll show that

$$P(k \text{ successes in } n \text{ trials}) = \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{\text{binomial dist}} \approx \underbrace{\frac{e^{-\lambda}\lambda^k}{k!}}_{\text{Poisson}}$$

So the Poisson can be used to approximate the binomial. The advantage of the Poisson, as you'll soon see, is that it has only the one parameter λ, while the binomial distribution has two parameters, n and p.

Here's why the approximation holds.

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k (1-p)^{n-k}$$

Substitute

$$p = \frac{\lambda}{n}$$

and rearrange to get

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{\lambda^k}{k!} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k} \qquad (2)$$

For large n, small p, moderate λ, and k much smaller than n,

$$\left(1 - \frac{\lambda}{n}\right)^n \sim e^{-\lambda} \quad \left(\text{remember that } \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a\right)$$

$$\left(1 - \frac{\lambda}{n}\right)^k = (1 - \text{small})^k \sim 1$$

$$\frac{n(n-1)(n-1)\cdots(n-k+1)}{n^k} = \frac{n^k + \text{lower degree terms}}{n^k} \sim 1$$

So (2) becomes

$$\binom{n}{k} p^k (1-p)^{n-k} \sim 1 \cdot \frac{\lambda^k}{k!}\frac{e^{-\lambda}}{1} = \frac{e^{-\lambda}\lambda^k}{k!}$$

Application of the Poisson to the Number of Successes in Bernoulli Trials

Your record as a typist shows that you make an average of 3 mistakes per page. I'll find the probability that you make 10 mistakes on page 437.

Each symbol typed is an experiment where the outcome is either error or OK. The symbols on a page are typed independently, so they are Bernoulli trials. The number of mistakes on a page has a binomial distribution, where n is the number of symbols on a page and p is the probability of an error in a symbol. But we don't know n or p, so we can't use the binomial distribution. On the other hand, we do know that

$$\text{average number of mistakes per page} = 3$$

so the next best choice is the Poisson approximation to the binomial, with $\lambda = 3$:

$$P(10 \text{ mistakes on page } 437) = \frac{e^{-3}\, 3^{10}}{10!}$$

Suppose you want the probability of fewer than 4 mistakes in the 10-page introduction. If the typist averages 3 mistakes per page, then on the average there are 30 mistakes in the introduction, so use the Poisson with $\lambda = 30$:

$$\begin{aligned} &P(\text{fewer than 4 mistakes in Intro}) \\ &= P(0 \text{ mistakes in Intro}) + P(1) + P(2) + P(3) \\ &= e^{-30}\left(1 + 30 + \frac{30^2}{2!} + \frac{30^3}{3!}\right) \quad (\text{Remember that } \lambda^0 = 1 \text{ and } 0! = 1.) \end{aligned}$$

Here's the general rule:

> Suppose you have a bunch of Bernoulli trials.
>
> You don't know n, the number of trials in the bunch, or the probability p of success on any one trial (if you did, you could use the binomial distribution).
>
> But you do know that the average number of successes in a bunch is λ.
>
> Then use the Poisson distribution to get
>
> $$P(k \text{ successes in a bunch}) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad (3)$$
>
> (provided that it's a large bunch and successes are fairly rare).

Example 1

The police ticket 5% of parked cars. (Assume cars are ticketed independently.) Find the probability of 1 ticket on a block with 7 parked cars.

Each car is a Bernoulli trial with $P(\text{ticket}) = .05$, so

$$P(1 \text{ ticket on block}) = P(1 \text{ ticket in 7 trials}) = \binom{7}{1}(.95)^6(.05)$$

Example 2

On the average, the police give out 2 tickets per block. Find the probability that a block gets 1 ticket.

The cars are Bernoulli trials. We don't know the number of cars on a block or $P(\text{ticket})$, but we can use the Poisson with $\lambda = 2$:

$$P(1 \text{ ticket on block}) = 2e^{-2}$$

Example 2 continued

Find the probability that a 4-block strip gets a least 5 tickets.

On the average, a 4-block strip gets 8 tickets, so use the Poisson with $\lambda = 8$:

$$\begin{aligned} &P(\text{at least 5 tickets on a 4-block strip}) \\ &= 1 - P(0) - P(1) - P(2) - P(3) - P(4) \\ &= 1 - e^{-8} - 8e^{-8} - \frac{8^2e^{-8}}{2!} - \frac{8^3e^{-8}}{3!} - \frac{8^4e^{-8}}{4!} \end{aligned}$$

Warning

When you use (3) to find the probability of k successes in *a bunch*, you must use as λ the average number of successes in *the bunch*. In example 2, for parking tickets in *a block*, use $\lambda = 2$, but for parking tickets in *a 4-block strip*, use $\lambda = 8$.

Application of the Poisson to the Number of Arrivals in a Time Period

As a telephone call arrives at a switchboard, the arrival time is noted and the switchboard is immediately ready to receive another call.

Let λ be the average number of calls in an hour, the rate at which calls arrive. Assume that calls arrive independently. We'll show why it's a good idea to use the Poisson as the model, that is, to use

$$(4) \qquad P(k \text{ calls in an hour}) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Divide the hour into a large number, n, of small time subintervals, so small that we can pretend that, at most, 1 call can arrive in a time subinterval. In other words, *during each subinterval, either no call arrives or 1 call arrives* (but it isn't possible for 2 calls to arrive). With this pretense, the n time subintervals are Bernoulli trials where success means that a call has arrived. We don't know n and p, but we do have λ, so it makes good sense to use the Poisson in (4).

Similarly, the Poisson distribution is the model for particles emitted, earthquakes occurring, and arrivals in general:

> If arrivals are independent, the number of arrivals in a time period has a Poisson distribution:
>
> $$(5) \qquad P(k \text{ arrivals in a time period}) = \frac{e^{-\lambda}\lambda^k}{k!}$$
>
> where the parameter λ is the arrival rate, the average number of arrivals in the time period.

Example 3

Suppose particles arrive on the average twice a second. Find the probability of at most 3 particles in the next 5 seconds.

The average number of particles in a 5-second period is 10, so use the Poisson distribution with $\lambda = 10$:

$$\begin{aligned} P(\text{at most 3 particles in 5 seconds}) &= P(\text{none}) + P(1) + P(2) + P(3) \\ &= e^{-10}\left(1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!}\right) \end{aligned}$$

Warning

The λ in example 3 must be 10, the average number of particles arriving per 5 *seconds*, not the original 2, which is the average number per second.

When you use (5) to find the probability of k arrivals in a time period, λ must be the average number of arrivals *in that time period*.

Problems for Section 2-5

1. On the average, a blood bank has 2 units of the rare type of blood, XYZ.
   - **(a)** Find the prob that a bank can supply at least 3 units of XYZ.
   - **(b)** If a community has two blood banks, find the prob that the community can supply at least 6 units of XYZ.

2. On the average there are 10 no-shows per airplane flight. If there are 5 flights scheduled, find the prob of
   (a) no no-shows (b) 4 no-shows (c) at most 4 no-shows
3. Assume drivers are independent.
   (a) If 5% of drivers fail to stop at the stop sign, find the prob that at least 2 of the next 100 drivers fail to stop.
   (b) If on the average 3 drivers fail to stop at the stop sign during each rush hour, find the prob that at least 2 fail to stop during tonight's rush hour.
4. On the average you get 2 speeding tickets a year.
   (a) Find the prob of getting 3 tickets this year.
   (b) Suppose you get 2 tickets in January. Find the prob of getting no tickets during the rest of the year (the other 11 months).
5. Phone messages come to your desk at the rate of 2 per hour. Find the prob that if you take a 15-minute break you will miss
   (a) no calls (b) no more than 1 call
6. On the average, in a year your town suffers through λ_1 earthquakes, λ_2 lightning strikes, and λ_3 meteorites crashing to earth. Find the prob that there will be at least one of these natural disasters next year.
7. On the average, you get 3 telephone calls a day. Find the prob that in 5 years there will be at least one day without a call. (This takes two steps. First, find the prob of no calls in a day.)
8. If $P(\text{H}) = .01$, then the prob of 1 H in 1000 tosses is $\binom{1000}{1}(.99)^{999}(.01)$. What's the Poisson approximation to this answer?
9. The binomial distribution, and to a lesser extent the Poisson distribution, involves Bernoulli trials. Do you remember what a Bernoulli trial is?

Review Problems for Chapters 1 and 2

1. Draw 10 balls from a box containing 20 white, 30 black, 40 red, and 50 green. Find each prob twice, once if the drawing is with replacement and again if it is without replacement.
   (a) $P(3\text{W and }4\text{R})$
   (b) $P(3\text{W followed by }4\text{R followed by 3 others})$
   (c) $P(4\text{R followed by }3\text{W followed by 3 others})$
2. Find the prob that among the first 9 digits from a random digit generator, there are at least four 2's.
3. Draw from a deck without replacement. Find the prob that
   (a) the 10th draw is a king and the 11th is a non-king

   (b) the first king occurs on the 10th draw
   (c) it takes 10 draws to get 3 kings
   (d) it takes at least 10 draws to get 3 kings
4. Find the prob that a bridge hand contains at least one card lower than 6 given that it contains at least one card over 9.
5. Form 12-symbol strings from the 26 letters and 10 digits. Find the prob that a string contains 3 vowels if repetition is
   (a) allowed (b) not allowed
6. If the letters in ILLINOIS are rearranged at random, find the prob that the permutation begins or ends with L. (Just as you can assume in a probability problem that white balls are named $W_1, \ldots, W_n$, you can assume here that the word is $I_1L_1L_2I_2NOI_3S$, with all the letters distinguishable.)
7. Find the prob that John and Mary are next to one another if eight people are seated at random
   (a) on a bench (b) around a circular table
8. Find the prob that a 3-letter word contains z (e.g., zzz, zab, bzc).
9. You notice that 1 out of every 10 cars parked in a tow-away zone is actually towed away. Suppose you park in a tow-away zone every day for a year. Find the prob that you are towed at most once.
10. At a banquet, m men and w women are introduced in random order to the audience. Find the prob that the last two introduced are men.
11. Given j married couples, k single men, and n single women, pick a man and a woman at random. Find the prob that
    (a) both are married
    (b) only one is married
    (c) they are married to each other
12. Find the prob of not getting any 3's when you toss a die
    (a) 10 times (b) 100,000 times (c) forever
13. Your drawer contains 5 black, 6 blue, and 7 white socks. Pull out 2 at random. Find the prob that they match.
14. If you get 6 heads and 4 tails in 10 tosses, find the prob that one of the heads was on the 8th toss.
15. Twenty-six ice cream flavors, A to Z, are available. Six orders are placed at random. Find the prob that the orders include
    (a) A and B once each
    (b) A and B at least once each
    (c) at least one of A and B (i.e., at least one A or at least one B)
    (d) two A's and at least two B's

(e) all different flavors
(f) all the same flavor

16. Consider the probability of getting a void in bridge, a hand with at least one suit missing.

(a) What's wrong with the following answer?

The total number of hands is $\binom{52}{13}$.

For the favorable hands:

Pick a suit to be missing. Can be done in 4 ways.

Pick 13 cards from the other 3 suits. Can be done in $\binom{39}{13}$ ways.

$$\text{Answer is } \frac{4\binom{39}{13}}{\binom{52}{13}}$$

(b) Find the right answer.

17. A box contains 6 black (named $B_1, \ldots, B_6$), 5 white, and 7 red balls. Draw balls. Find the prob that

(a) B_3 is drawn before B_5
(b) B_3 is drawn before any of the whites

18. One IRS office has three people to answer questions. Mr. X answers incorrectly 23% of the time, Ms. Y 3% of the time, and Ms. Z 4% of the time. Of all questions directed to this office, 60% are handled by X, 30% by Y, and 10% by Z.

What percentage of incorrect answers given by the office is due to Z?

19. Toss a coin 10 times. Find the prob of getting no more than 5 heads given that there are at least 3 heads.

20. Call the throw of a pair of dice lucky if the sum is 7 or 11.

Two players each toss a pair of dice (independently of one another) until each makes a lucky throw. Find the prob that they take the same number of throws.

21. A basketball player has made 85% of her foul shots so far in her career. Find the prob that she will make at least 85% of her next 10 foul shots.

22. Prizes are given out at random in a group of people. It's possible for a person to get more than one prize.

(a) If there are 10 prizes and 5 people, find the prob that no one ends up empty-handed.
(b) If there are 5 prizes and 10 people, find the prob that no one gets two (or more) prizes.
(c) If there are 5 prizes and 6 men and 4 women, find the prob that all the prizes go to men.

23. Toss a penny and a nickel 20 times each. For each coin, $P(\text{H}) = .7$. If the overall result is 17H and 23T, find the prob that 11 of those 17 heads were from the penny.

24. Four hundred leaflets are dropped at random over 50 square blocks.

(a) Find the prob that your block gets at least 3.
(b) Find the Poisson approximation to the answer in (a).

25. Three players toss coins simultaneously. For each player, $P(\text{H}) = p$, $P(\text{T}) = q$. If the result is 2H and 1T or the result is 2T and 1H, then the player that is different from the other two is called the odd man out and the game is over. If the result is 3H or 3T, then the players toss again until they get an odd man out.

Find the prob that the game lasts at least 6 rounds.

26. Let $P(A) = .5$, $P(B) = .2$, $P(C) = 1$. Find $P(A \text{ or } B \text{ or } C)$ if

(a) A, B, C are mutually exclusive
(b) A, B, C are independent

27. On the average, there is a power failure once every four months.

(a) Find the prob of a power failure during exam week.
(b) Find the prob that it will be at least a month before the next failure.

28. A bus makes 12 stops and no one stop is more popular than another. If 5 passengers travel on the bus independently, what's the prob that 3 get off at one stop and 2 get off at another (so that you have a full house of stops).

29. Find (a) $\binom{999}{1}$ (b) $\binom{999}{0}$ (c) $\binom{1000}{999}$ (d) $\dfrac{\binom{n+m-1}{n-1}}{\binom{n+m}{n}}$

30. Teams A and B meet in the world series (the first to win 4 games is the series winner). Assume the teams are evenly matched and the games are independent events. Find the prob that the series ends in 6 games.

31. Five shots are fired at random into a circle of radius R. The diagram shows an inscribed square and four other zones. Find the prob that the 5 shots end up

(a) all in the same zone
(b) in five different zones

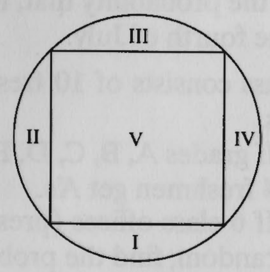


Figure P.31

32. Mary Smith has a 50-50 chance of carrying an XYZ gene. If she is a carrier, then any child has a 50-50 chance of inheriting the gene. Find the prob that her 4th child will not have the gene given that her first three children don't have it.

33. (the famous birthday problem) Find the prob that in a group of n people, at least two will have the same birthday.

34. You have 5 dice and 3 chances with each die to get a 6.

For example, if you toss the third die and get a 6, then you move on to the fourth die. But if the third die is non-6, you get to try again and then again if necessary.

Find the prob of getting two 6's overall with the 5 dice.

Suggestion: First find the prob of getting a 6 from a single die in your 3 chances.

35. A message is sent across a channel to a receiver.

The probability is .6 that the message is $xxxxx$.
The probability is .4 that the message is $yyyyy$.

For each letter transmitted, the probability of error (that an x will become y, or vice versa) is .1. Find the probability that the message was $xxxxx$ if 2 x's and 3 y's are received.

36. John's score is a number chosen at random between 0 and 3. Mary's score is chosen at random between 0 and 1. The two scores are chosen independently. Find the prob that

(a) his score is at least twice hers
(b) the max of the two scores is $\leq 1/2$
(c) the min of the two scores is $\geq 1/2$

37. Five people are picked (without replacement) from a group of 20. Find the prob that John was chosen using these different methods.

(a) fav committees/total committees
(b) P(John was chosen 1st or 2nd or ... or chosen 5th)
(c) $1 - P$(no John)

38. Find the probability that, in a group of 30 people, at least 3 were born on the fourth of July.

39. A class consists of 10 freshmen, 20 sophomores, 30 juniors, and 20 grads.

(a) If grades A, B, C, D, E are assigned at random, find the prob that 4 freshmen get A's.
(b) If 6 class offices (president, vice president, etc.) are assigned at random, find the prob that 4 freshmen get offices.

40. (*Computer Networks*, Tannenbaun, Prentice Hall, 1989) A disadvantage of a broadcast subnet is the capacity wasted due to multiple hosts attempting to access the channel at the same time. Suppose a time period is divided into a certain number of discrete slots. During each time slot, the probability is p that a host will want to use the channel. If two or more hosts want to use a time slot, then a collision occurs, and the slot is wasted. If there are n hosts, what fraction of the slots is wasted due to collisions?

41. John will walk past a street corner some time between 10:00 and 11:00. Mary will pass the same street corner some time between 10:00 and 11:30. Find the prob that they meet at the corner if

(a) each agrees to wait 10 minutes for the other
(b) John will wait 10 (lovesick) minutes for Mary (but not vice versa)

42. Draw 20 times from the integers 1, 2, 3, ..., 100. Find the prob that your draws come out in increasing order (each draw is larger than the previous draw), if the drawing is

(a) with replacement **(b)** without replacement

You can do it directly with fav/total (but most people get stuck on the fav).

The solution to

$$\frac{16}{c^2} = .1$$

is $c = 4\sqrt{10}$. So

$$P(X - 28 \geq 4\sqrt{10}) \leq .1$$
$$P(X \geq 28 + 4\sqrt{10}) \leq .1$$

If you make $28 + 4\sqrt{10} \approx 41$ items, you'll meet the demand at least 90% of the time.

In this example, Chebychev's inequality gave a better answer than Markov's inequality. (And someone else's inequality may come along and give a still better answer.)

Review Problems for Chapters 4–9

1. Let X and Y have joint density $f(x, y)$ with universe $0 \leq y \leq 3, x \geq y$. Set up the integrals for the following.
   - (a) $P(XY \leq 2)$
   - (b) the distribution of $X - Y$
   - (c) $P(X \leq 2|Y = y)$
   - (d) EX
   - (e) $E(Y|X = x)$
   - (f) the distribution of the max
   - (g) $E(\max)$
2. Let X be uniform on $[0, 1]$. Find the distribution of $1/(X + 1)$.
3. The lifetime of a bulb (measured in days) has mean 10.2 and variance 9. When a bulb burns out, it is replaced by a similar bulb. Find the prob that more than 100 bulbs are needed in the next 3 years.

Solutions

Solutions Section 1-1

1. There are 6 favorable outcomes, (3,4), (4,3), (6,1), (1,6), (5,2), (2,5), so prob is 6/36.
2. 8/36
3. There are 6 outcomes where the two dice are equal. Of the remaining 30 outcomes, half have second > first. Answer is 15/36.
4. Fav outcomes are (6,1), ..., (6,6), (1,6), ..., (6,6), but don't count (6,6) twice. Answer is 11/36.
5. Fav outcomes are (5,5), (5,6), (6,5), (6,6). Answer is 4/36.
6. The fav outcomes lie in columns 5 and 6 and in rows 5 and 6 in (4). Prob is 20/36.
7. $P(\text{neither over } 4) = 1 - P(\text{at least one} \geq 5)$
   $= 1 -$ answer to problem 6 $= 16/36$.
8. 9/36
9. $P(\text{at least one odd}) = 1 - P(\text{both even}) = 1 - \dfrac{9}{36} = \dfrac{27}{36}$

Solutions Section 1-2

1. (a) $\dfrac{\binom{8}{4} \text{ pick 4 others}}{\binom{9}{5} \text{ total}}$ (b) $\dfrac{\binom{7}{5} \text{ pick 5 from the others}}{\binom{9}{5}}$
2. For the favs, pick 9 more cards from the 39 non-spades. Prob is $\binom{39}{9}/\binom{52}{13}$.

3. $P(\text{not both C}) = 1 - P(\text{both C}) = 1 - \dfrac{\binom{7}{2} \text{ pick 2 from the 7 C's}}{\binom{18}{2} \text{ pick 2 people}}$

4. **(a)** $\dfrac{\binom{39}{5}}{\binom{52}{5}}$ **(b)** $\dfrac{\binom{50}{3}}{\binom{52}{5}}$ **(c)** $\dfrac{\binom{48}{3}}{\binom{52}{5}}$

5. **(a)** The total number of ways in which the coats can be returned is 4! (think of each woman as a slot; the first can get any of 4 coats, the second any of 3 coats, etc.). Only one way is fav, so prob is 1/4!.

(b) Fill Mary's slot. Total number of ways is 4, fav is 1. Answer is 1/4.

6. **(a)** $7 \cdot 6 = 42$ **(b)** $\dfrac{7!}{2!\,5!} = \dfrac{7 \cdot 6}{1 \cdot 2} = 21$

(c) $\dfrac{8!}{5!\,3!} \Big/ \dfrac{4!}{3!\,1!} = \dfrac{8!\,3!}{5!\,3!\,4!} = \dfrac{8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 14$

7. Assume John has chosen his name. Find the prob that Mary makes the same choice. When Mary picks her name, only one is favorable (John's choice).

To count the total number of names, consider names of length 1, of length 2, ..., of length 7 and add.

For names of length 1, there is one spot to fill. It must be done with a letter, so there are 26 possibilities.

For names of length 2 there are two spots to fill. The first must be a letter, and the second can be any of 36 symbols (26 letters, 10 digits). So there are $26 \cdot 36$ possibilities.

For names of length 3, there are 3 spots to fill. The first must be a letter, and the next two spots can be any of 36 symbols. So there are $26 \cdot 36^2$ names of length 3.

And so on.

All in all,

$P(\text{same name})$

$$= \frac{1}{26 + 26 \cdot 36 + 26 \cdot 36^2 + 26 \cdot 36^3 + 26 \cdot 36^4 + 26 \cdot 36^5 + 26 \cdot 36^6}$$

Why is there a *sum* in the denominator? Because the *total* number of names is the number of names of length 1 *plus* the number of names of length 2 *plus* the number of names of length 3 *plus*, and so on.

8. **(a)** $7 \cdot 7 \cdot 7$ **(b)** $7 \cdot 6 \cdot 5$ **(c)** $\dbinom{7}{3}$

(d) With replacement, unordered, that is, a committee where someone can serve more than once (a histogram).

For example, $A_2A_2A_5$ is one such sample and is the *same* as $A_2A_5A_2$.

9. There are 4 fav outcomes (spade flush, heart flush, diamond, clubs). Answer is $4/\binom{52}{5}$.

10. **(a)** Total number of outcomes is 7^3 (each person is a slot). There are 7 favorable outcomes (all go to church 1, all go to church 2, etc.). Answer is $7/7^3 = 1/49$.

(b) $1 -$ answer to (a) $= 48/49$.

(c) $(7 \cdot 6 \cdot 5)/(7 \cdot 7 \cdot 7)$.

(d) $1 -$ answer to (c) $= 1 - (7 \cdot 6 \cdot 5)/7^3$.

(e) The total number of draws is $\binom{54}{6}$ and the favorable number is 2. Answer is $2/\binom{54}{6}$, which is approximately 1/13,000,000.

Solutions Section 1-3

1. **(a)** $\dfrac{\binom{13}{3}\binom{13}{2} \ \text{pick 3 diamonds, pick 2 hearts}}{\binom{52}{5}}$ **(b)** $\dfrac{12 \cdot \binom{39}{3} \ \text{pick non-ace spade, pick 3 non-spades}}{\binom{52}{5}}$

(c) $\dfrac{\binom{26}{4} \cdot 26}{\binom{52}{5}}$ **(d)** $\dfrac{\binom{4}{2}\binom{48}{3} \ \text{pick 2 aces, pick 3 non-aces}}{\binom{52}{5}}$

(e) $\dfrac{\binom{50}{4} \ \text{pick 4 more cards to go with the spade ace but don't pick spade king}}{\binom{52}{5}}$

2. **(a)** $\dfrac{\binom{3}{2}\binom{14}{2} \ \text{pick 2 A's, pick 2 non-A's non-}R_3\text{'s}}{\binom{18}{5}}$ **(b)** $\dfrac{8 \cdot \binom{10}{4}}{\binom{18}{5}}$

3. Total is $\binom{8}{3}$. For the fav, pick 3 couples. Then pick one spouse from each couple.

$$\frac{\binom{4}{3} \cdot 2^3}{\binom{8}{3}}$$

4. For the total, pick a committee of 12 symbols from the 36. For the fav, pick a subcommittee of 3 evens and a subcommittee of 9 others.

$$\frac{\binom{5}{3}\binom{31}{9}}{\binom{36}{12}}$$

5. $$\frac{\binom{25}{3}\binom{30}{2}\binom{90}{6}}{\binom{145}{11}}$$

6. There are $\binom{7}{4}$ ways of picking 4 seats. Only 4 ways have the seats together (namely, the blocks S_1–S_4, S_2–S_5, S_3–S_6, S_4–S_7). Answer is $4/\binom{7}{4}$.

7. $$\frac{13\binom{4}{3}\ \begin{matrix}\text{pick the face value}\\ \text{pick 3 cards from that face}\end{matrix}}{\binom{52}{3}}$$

8. **(a)** $$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\ \begin{matrix}\text{pick 2 faces for the pairs}\\ \text{pick 2 cards from each face}\end{matrix}}{\binom{52}{4}}$$

(b) For the fav, pick a face for the 3 of a kind and then a face for the pair. Then pick 3 cards from the first face and 2 from the second face. Answer is

$$\frac{13 \cdot 12\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

Why did part (a) have $\binom{13}{2}$ in the numerator while part (b) has $13 \cdot 12$? In (b) there are two slots: the face-for-the-3-of-a-kind and the face-for-the-pair. In (a) we need 2 faces but they are not slots because there isn't a first pair and a second pair—we want a committee of 2 faces.

9. **(a)** $$\frac{4\binom{13}{5}\ \begin{matrix}\text{pick a suit}\\ \text{pick 5 cards from that suit}\end{matrix}}{\binom{52}{5}}$$

(b) $$\frac{48\ \text{pick one remaining card}}{\binom{52}{5}}$$

(c) $$\frac{13 \cdot 48\ \begin{matrix}\text{pick a face value for the four}\\ \text{pick a fifth card}\end{matrix}}{\binom{52}{5}}$$

(d) For the fav, pick a face value for the pair and then pick 2 cards in that face. To make sure you don't get any more of that face value and don't get any other matching faces (i.e., to avoid a full house, 2 pairs, 3 or 4 of a kind), pick 3 other faces and pick a card from each of those faces. Answer is

$$\frac{13 \cdot \binom{4}{2}\binom{12}{3} \cdot 4^3}{\binom{52}{5}}$$

10. Lining up objects is the same as drawing without replacement. By symmetry,

$$P(\text{girl in the } i\text{th spot} = P(\text{girl in the first spot}) = \frac{g}{b+g}$$

11. **(a)** This method counts the outcomes J_S, J_H, J_C, A_S, 3_H and J_S, J_H, J_C, 3_H, A_S as different when they are the same hand. In general this method counts every outcome twice.
Here's the correct version.

Pick a face value and 3 cards from that value.
Pick 2 more faces from the remaining 12 (a *committee* of 2 faces).
Pick a card in each of those 2 faces.

Answer is $13 \cdot \binom{4}{3}\binom{12}{2} \cdot 4^2$.
So the prob of 3 of a kind is

$$\frac{13 \cdot \binom{4}{3}\binom{12}{2} \cdot 4^2}{\binom{52}{5}}$$

(b) It counts the following outcomes as different when they are really the same.

outcome 1	**outcome 2**
Pick spot 1 for an A.	Pick spot 2 for an A.
Pick spot 2 to get an A.	Pick spot 1 for an A.
Pick spot 7 to get an A.	Pick spot 7 for an A.
Fill other spots with Z's.	Fill other spots with Z's.

Here's the correct version.

Pick a committee of 3 spots for the A's.
Fill the other places from the remaining 25 letters.

Answer is $\binom{7}{3} \cdot 25^4$.
So the prob of 3 A's in a 7-letter word is

$$\frac{\binom{7}{3} \cdot 25^4}{26^7}$$

(c) It counts the hand $J_H\,3_S$ as different from the hand $3_S\,J_H$. In fact it counts every outcome exactly twice. It uses "first card" and "second card" as slots, but there is no such thing as a first or a second card in a hand.

Here's the correct version. Pick a committee of 2 faces. Pick a card from each face. Answer is $\binom{13}{2}\cdot 4^2$.

So the prob of a pair is

$$\frac{\binom{13}{2}4^2}{\binom{52}{2}}$$

Solutions Section 1-4

1. (a) $P(2A)+P(2K)-P(2A \text{ and } 2K) = \dfrac{\binom{4}{2}\binom{48}{3}+\binom{4}{2}\binom{48}{3}-\binom{4}{2}\binom{4}{2}\cdot 44}{\binom{52}{5}}$

(b) (mutually exclusive events) $P(3A)+P(3K) = \dfrac{\binom{4}{3}\binom{48}{2}+\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$

(c) *Method 1.*

$$P(A_S)+P(K_S)-P(A_S \text{ and } K_S) = \frac{\binom{51}{4}+\binom{51}{4}-\binom{50}{3}}{\binom{52}{5}}$$

Method 2.

$$1-P(\text{not spade ace and not spade king}) = 1-\frac{\binom{50}{5}}{\binom{52}{5}}$$

2. (a) $P(3W)+P(2R)+P(5G)-P(3W \text{ and } 2R)$

$\left(\begin{array}{l}\text{Can leave out the rest of the terms since}\\ \text{an event such as "3W and 5G" is impossible.}\end{array}\right)$

$$= \frac{\binom{10}{3}\binom{50}{2}+\binom{20}{2}\binom{40}{3}+\binom{30}{5}-\binom{10}{3}\binom{20}{2}}{\binom{60}{5}}$$

(b) $P(5W \text{ or } 5R \text{ or } 5G)$

$$= P(5W)+P(5R)+P(5G) = \frac{\binom{10}{5}+\binom{20}{5}+\binom{30}{5}}{\binom{60}{5}}$$

3. (a) Counting these terms is like counting twosomes from a population of size 8. There are $\binom{8}{2}$ such terms.

(b) Count committees of size 3 from a pop of 8. There are $\binom{8}{3}$ such terms.

4. (a) $P(\text{no women})+P(\text{no H})-P(\text{no W and no H})$

$$= \frac{\binom{17}{12}+\binom{37}{12}-\binom{15}{12}}{\binom{42}{12}}$$

(b) $P(\text{only non-H men}) = \dfrac{\binom{15}{12}}{\binom{42}{12}}$

5. *Method 1.* XOR = OR − BOTH, so

$$P(\text{J XOR Q}) = P(\text{J OR Q}) - P(\text{both})$$

$$= P(J)+P(Q)-P(\text{J and Q})-P(\text{J and Q}) \text{ again}$$

$$= \frac{\binom{51}{4}+\binom{51}{4}-2\binom{50}{3}}{\binom{52}{5}}$$

Method 2.

$$P(\text{J and not Q})+P(\text{Q and not J}) = \frac{\binom{50}{4}+\binom{50}{4}}{\binom{52}{5}}$$

6. (a) $1-P(\text{no S}) = 1-\dfrac{\binom{39}{5}}{\binom{52}{5}}$

(b) *Method 1.*

$$P(3S \text{ or } 4S \text{ or } 5S) = P(3S)+P(4S)+P(5S)$$

$$= \frac{\binom{13}{3}\binom{39}{2}+\binom{13}{4}\cdot 39+\binom{13}{5}}{\binom{52}{5}}$$

Method 2.

$$1-[P(\text{no S})+P(1S)+P(2S)] = 1-\frac{\binom{39}{5}+13\binom{39}{4}+\binom{13}{2}\binom{39}{3}}{\binom{52}{5}}$$

(c) *Method 1.*

$$1 - P(3A) - P(4A) = 1 - \frac{\binom{4}{3}\binom{48}{2} + 48}{\binom{52}{5}}$$

Method 2.

$$P(\text{no A}) + P(1A) + P(2A) = \frac{\binom{48}{5} + 4\binom{48}{4} + \binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$$

(d) *Method 1.*

$$P(4 \text{ pics}) - P(4 \text{ pics with no aces}) - P(4 \text{ pics with 1 ace})$$

$$= \frac{\binom{16}{4}36 - \binom{12}{4}36 - 4\binom{12}{3}36}{\binom{52}{5}}$$

Method 2.

$$P(4 \text{ pics with 2 aces}) + P(4 \text{ pics with 3 aces})$$

$$+ P(4 \text{ pics with 4 aces}) = \frac{\binom{4}{2}\binom{12}{2}36 + \binom{4}{3}\cdot 12\cdot 36 + 36}{\binom{52}{5}}$$

7. (a) $P(\text{spades or hearts or diamonds or clubs})$

$$= P(S) + P(H) + P(D) + P(C) - \underbrace{[P(S \text{ and } H) + \cdots]}_{\binom{4}{2}\text{terms in here}}$$

(no room for 3 or more royal flushes)

$$= \frac{4\binom{47}{8} - \binom{4}{2}\binom{42}{3}}{\binom{52}{13}}$$

(b) $P(4\ 2\text{'s or } 4\ 3\text{'s or} \ldots \text{ or } 4 \text{ aces})$

$$= P(4\ 2\text{'s}) + \cdots + P(4 \text{ aces})$$

$$-\underbrace{[P(4\ 2\text{'s and } 4\ 3\text{'s}) + \text{other 2-at-a-time-terms}]}_{\binom{13}{2} \text{ terms, each is } \binom{44}{5}\Big/\binom{52}{13}}$$

$$+\underbrace{[P(4\ 2\text{'s and } 4\ 3\text{'s and } 4\ J\text{'s}) + \cdots + \text{other 3-at-a-time-terms}]}_{\binom{13}{3} \text{ terms, each is } 40/\binom{52}{13}}$$

$$= \frac{13\binom{48}{9} - \binom{13}{2}\binom{44}{5} + \binom{13}{3}40}{\binom{52}{13}}$$

8. *Method 1.*

$$P(\text{happy Smith family}) = 1 - P(\text{no Smiths win}) = 1 - \frac{\binom{96}{3}}{\binom{100}{3}}$$

Method 2.

$$P(\text{happy Smiths}) = P(\text{1 Smith wins or 2 Smiths win or 3 Smiths win or 4 Smiths win})$$

$$= P(1S) + P(2S) + P(3S) + P(4S)$$

To compute $N(\text{1 Smith wins})$, pick the Smith in 4 ways, the other 2 winners in $\binom{96}{2}$ ways.

To compute $N(\text{2 Smiths win})$, pick the 2 Smiths in $\binom{4}{2}$ ways, the other winner in 96 ways.

To compute $N(\text{3 Smiths win})$, pick the 3 Smiths in $\binom{4}{3}$ ways. So

$$P(\text{happy Smith family}) = \frac{4\binom{96}{2} + 96\binom{4}{2} + \binom{4}{3}}{\binom{100}{3}}$$

Method 3.

$$P(\text{happy Smith family}) = P(\text{John wins or Mary wins or Bill wins or Henry wins})$$

$$= P(\text{John}) + P(\text{Mary}) + P(\text{Bill}) + P(\text{Henry}) \quad \left(4 \text{ terms, each is } \binom{99}{2}\Big/\binom{100}{3}\right)$$

$$- [P(\text{JohnMary}) + P(\text{JohnBill}) + \cdots] \quad \left(\binom{4}{2} \text{ terms, each is } 98\Big/\binom{100}{3}\right)$$

$$+ [P(\text{JMB}) + P(\text{JMH}) + \cdots] \quad \left(\binom{4}{3} \text{ terms, each is } 1\Big/\binom{100}{3}\right)$$

$$- P(\text{JMBH}) \quad \text{(this term is 0 since there are only 3 winners)}$$

$$= \frac{4\binom{99}{2} - \binom{4}{2}98 + \binom{4}{3}}{\binom{100}{3}}$$

9. **(a)** $\dfrac{7\binom{11}{3}}{\binom{18}{4}}$

(b) $1 - P(\text{no women}) = 1 - \binom{11}{4}\Big/\binom{18}{4}$

(c) $P(\text{no women}) + P(1 \text{ woman}) = \dfrac{\binom{11}{4}+7\binom{11}{3}}{\binom{18}{4}}$

(d) $1 - P(\text{no M or no W or no C})$

$$= 1 - \left[\begin{array}{l} P(\text{no M}) + P(\text{no W}) + P(\text{no C}) \\ -[P(\text{no M \& no W}) + \cdots] \end{array}\right]$$

$$= 1 - \frac{\binom{12}{4}+\binom{11}{4}+\binom{13}{4}-\left[\binom{5}{4}+\binom{6}{4}+\binom{7}{4}\right]}{\binom{18}{4}}$$

(e) $P(\text{no women}) - P(\text{no women and no men}) = \dfrac{\binom{11}{4}-\binom{5}{4}}{\binom{18}{4}}$

10. **(a)** $1 - P(\text{no H or no M or no P})$

$$= 1 - \left[\begin{array}{l} P(\text{no H}) + P(\text{no M}) + P(\text{no P}) \\ -[P(\text{no H, no M}) + P(\text{no H, no P}) + P(\text{no M, no P})] \\ +P(\text{no H, no M, no P}) \end{array}\right]$$

$$= 1 - \frac{3\binom{98}{15} - 3\binom{96}{15} + \binom{94}{15}}{\binom{100}{15}}$$

(b) $1 - P(\text{no H and no M and no P}) = 1 - \binom{94}{15}\Big/\binom{100}{15}$

(c) *Method 1.*

$$P(1\text{H}) - P(1\text{H and no M}) = \frac{2\binom{98}{14} - 2\binom{96}{14}}{\binom{100}{15}}$$

Method 2.

$$P(1\text{H and 1M}) + P(1\text{H and 2M}) = \frac{2 \cdot 2\binom{96}{13} + 2\binom{96}{12}}{\binom{100}{15}}$$

11. **(a)** Think of the husbands as slots. The total number of ways they can be filled is 7! For the favorable, fill H_3 in one way, the others in 6! ways. Answer is $6!/7! = 1/7$.

(You can get this immediately by considering the prob that the H_3 slot gets the 1 favorable out of 7 total possibilities.)

(b) For the favorable: The H_2, H_5, H_7 slots are determined. Fill the others in 4! ways. Answer is $4!/7! = 1/(7 \cdot 6 \cdot 5)$.

(c) $P(H_1 \text{ or } H_2 \text{ or } \ldots \text{ or } H_7 \text{ is matched with his wife})$

$$= P(H_1) + \cdots + P(H_7) - [P(H_1 \,\&\, H_2) + \cdots]$$

$$+ \left(\text{3-at-a-time-terms—there are } \binom{7}{3} \text{ of them and by part (b) each is } 4!/7!\right)$$

$$- (\text{4-at-a-time-terms}) + (\text{5-at-a-time-terms}) - (\text{6-at-a-time}) + P(\text{all match})$$

$$= 7\frac{6!}{7!} - \binom{7}{2}\frac{5!}{7!} + \binom{7}{3}\frac{4!}{7!} - \binom{7}{4}\frac{3!}{7!} + \binom{7}{5}\frac{2!}{7!} - \binom{7}{6}\frac{1!}{7!}$$

$$+ \binom{7}{7}\frac{0!}{7!} \quad \text{(the last term is 1/7!, the prob they all match)}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!}$$

(d) 1 − answer to (c)

Solutions Section 1-5

1. **(a)** and **(b)** fav length/total length $= \frac{1}{2}/2 = 1/4$

(c) $.2/2 = 1/10$

(d) To find the sol to $3x^2 > x$, first solve $3x^2 = x$ to get $x = 0$, $1/3$. Then look in between at intervals $(-1,0)$, $(0,\frac{1}{3})$, $(\frac{1}{3},1)$ to see where $3x^2 > x$. The inequality is satisfied in $(-1,0)$ and $(\frac{1}{3}, 1)$. So fav/total $= (5/3)/2 = 5/6$.

2. $\dfrac{\text{fav area}}{\text{total area}} = \dfrac{4\pi}{81\pi} = \dfrac{4}{81}$

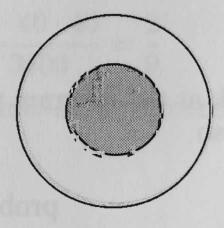


Figure P.2

3. The roots of $ax^2 + bx + c = 0$ are real iff $b^2 - 4ac \geq 0$, so in this case we need

$$16Q^2 - 4 \cdot 4(Q + 2) \geq 0$$
$$Q^2 - Q - 2 \geq 0$$
$$Q \geq 2 \text{ or } Q \leq -1$$

So the fav Q's in [0,5] are $2 \leq Q \leq 5$ and the answer is fav/total = 3/5.

4. $\sin\theta > \frac{1}{3}$ where $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$ iff $\theta > \sin^{-1}\frac{1}{3}$

$$\text{prob} = \frac{\text{fav length}}{\text{total}} = \frac{\frac{1}{2}\pi - \sin^{-1}\frac{1}{3}}{\pi}$$

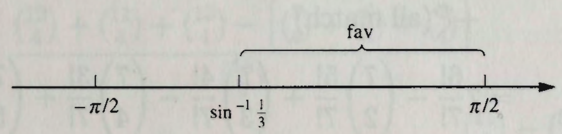


Figure P.4

5. (a) The diagram shows several chords and a conveniently placed equilateral triangle for comparison. The favorable θ's, that correspond to long enough chords, are between 60° and 120°, so

$$\text{prob} = \frac{\text{fav length}}{\text{total}} = \frac{60}{180} = \frac{1}{3}$$

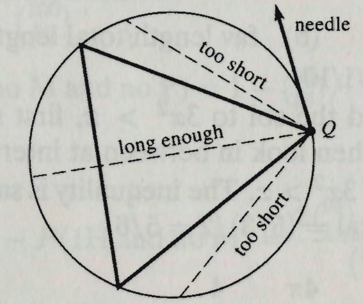


Figure P.5a

(b) Look at the diagram to see that favorable d's lie between 0 and $\frac{1}{2}R$, so

$$\text{prob} = \frac{\text{fav length}}{\text{total}} = \frac{\frac{1}{2}R}{R} = \frac{1}{2}$$

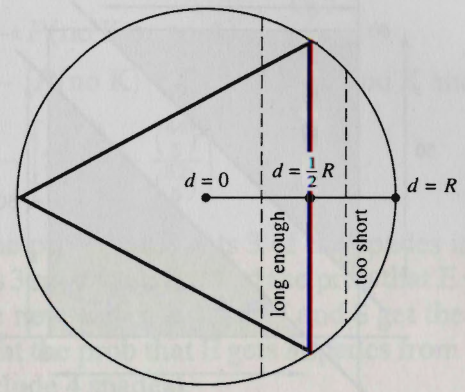


Figure P.5b

6. Let x be the first number; let y be the second number. Then the pair (x, y) is uniformly distributed in the rectangle $0 \leq x \leq 1, 1 \leq y \leq 3$.

(a) $P(x + y \leq 3) = \dfrac{\text{fav area}}{\text{total}} = 1 - \dfrac{\text{unfav}}{\text{total}} = 1 - \dfrac{1/2}{2} = \dfrac{3}{4}$

(b) total = 2, fav $= \displaystyle\int_{x=1/3}^{1} \left(3 - \frac{1}{x}\right) dx = 2 + \ln\tfrac{1}{3} = 2 - \ln 3$

$P(xy > 1) = 1 - \frac{1}{2}\ln 3$

7. (a) Let x be John's arrival time and let y be Mary's arrival time. Then (x, y) is uniformly distributed in the indicated square and

$$P(\text{first to arrive must wait at least 10 minutes for the other})$$
$$= P(|x - y| > 10) \quad \text{(i.e., the arrival times differ by more than 10)}$$
$$= \frac{\text{fav area}}{\text{total}} = \frac{2500}{3600} = \frac{25}{36}$$

(b) $P(y - x \geq 20) = \dfrac{\text{fav}}{\text{total}} = \dfrac{\frac{1}{2} \cdot 40 \cdot 40}{3600} = \dfrac{2}{9}$

(c) The event consists of all points on a segment. The fav *area* is 0. $P(y = x) = 0$.

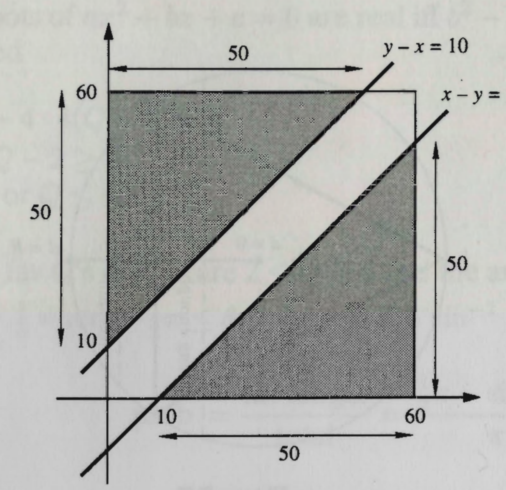


Figure P.7a

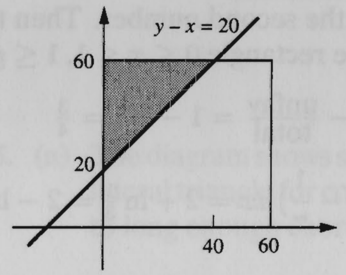


Figure P.7b

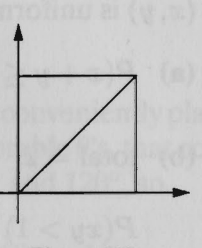


Figure P.7c

Solutions Section 2-1

1. (a) 3/51 (b) 4/51
   (c) $P(\text{2nd is king or lower}|\text{1st is K}) = 47/51$
   (d) $P(\text{second is non-ace}|\text{first is ace}) = 48/51$

2. (a) $1 - P(\text{rain}|\text{Jan 7}) = .7$
   (b) Can't do with the information given.

3. The new universe of 8-sums contains the five equally likely points (4,4), (5,3), (3,5), (6,2), (2,6). So $P(\text{first die is 6}) = 1/5$.

4. $P(\text{at least one K}|\text{at least one A})$

$$= \frac{P(\text{at least one K and at least one A})}{P(\text{at least one A})}$$

$$\text{denominator} = 1 - P(\text{no aces}) = 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

$$\text{numerator} = 1 - P(\text{no K or no A})$$

$$= 1 - [P(\text{no K}) + P(\text{no A}) - P(\text{no K and no A})]$$

$$= 1 - \left[2\frac{\binom{48}{5}}{\binom{52}{5}} - \frac{\binom{44}{5}}{\binom{52}{5}}\right]$$

5. By symmetry, the prob that E gets 3 of the spades is the same as the prob that W gets 3 spades, so just find the prob that E gets 3 spades and double it. In the new universe (after N and S get their hands with the 9 spades) we want the prob that E gets 3 spades from the 26 remaining cards (which include 4 spades).

$$P(\text{3–1 split}) = 2P(E \text{ gets 3 spades})$$

$$= \frac{\binom{4}{3}\binom{22}{10} \begin{array}{l}\text{pick 3 spades}\\ \text{pick 10 others}\end{array}}{\binom{26}{13}}$$

6. *Method 1.*

$$P(ABD|ABC) = \frac{P(\text{in } ABD \text{ and in } ABC)}{P(\text{in } ABC)}$$

$$\text{numerator} = \frac{\text{fav area}}{\text{total}} = \frac{\text{area } ABE}{1} = \frac{1}{4}$$

$$\text{denom} = \frac{\text{fav area}}{\text{total}} = \frac{1/2}{1} = \frac{1}{2}$$

$$\text{Answer} = \frac{1/4}{1/2} = \frac{1}{2}$$

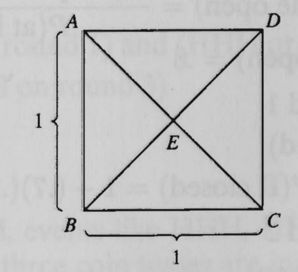


Figure P.6

Method 2. In the new universe ABC,

$$P(\text{ in } ABC) = \frac{\text{fav area}}{\text{total}} = \frac{\text{area } ABE}{\text{area } ABC} = \frac{1/4}{1/2} = \frac{1}{2}$$

7. **(a)** $P(\text{B R B W}) = P(\text{B})P(\text{R}|\text{B})P(\text{B}|\text{BR})P(\text{W}|\text{BRB}) = \frac{9}{24}\frac{5}{23}\frac{8}{22}\frac{10}{21}$

(b) Same as (a) by symmetry.

(c) $\frac{9}{24}\frac{5}{23}\frac{8}{24}\frac{10}{23}$

(d) $\frac{\binom{9}{2}\cdot 5\cdot 10}{\binom{24}{4}}$

(e) By symmetry, $P(\text{W on 4th}) = P(\text{W on 1st}) = 10/24$.

(f) By symmetry,

$$P(\text{W on 3rd, W on 4th}) = P(\text{W on 1st, W on 2nd}) = \frac{10}{24}\frac{9}{23}$$

8. $P(\text{saved}) = P(\text{at least one message gets through})$.

Method 1.

$$1 - P(\text{no smoke and no bottle and no pigeon})$$
$$= 1 - P(\text{no s})P(\text{no b})P(\text{no p}) = 1 - (.9)(.8)(.7)$$

Method 2.

$$P(\text{smoke or bottle or pigeon})$$
$$= P(\text{s}) + P(\text{b}) + P(\text{p})$$
$$-[P(\text{s and b}) + P(\text{s and p}) + P(\text{b and p})] + P(\text{sbp})$$
$$= .1 + .2 + .3 - [(.1)(.2) + (.1)(.3) + (.2)(.3)] + (.1)(.2)(.3)$$

9. $P(\text{II open}|\text{at least one open}) = \frac{P(\text{II open and at least one open})}{P(\text{at least one open})}$

numerator $= P(\text{II open}) = .8$

denominator method 1

$= 1 - P(\text{both closed})$

$= 1 - P(\text{I closed})P(\text{II closed}) = 1 - (.7)(.2)$

denominator method 2

$= P(\text{I open or II open})$

$= P(\text{I open}) + P(\text{II open}) - P(\text{I and II open})$

$= .3 + .8 - (.3)(.8)$

10. **(a)** The interpretation is that we want $P(\text{2B}|\text{at least one B})$.

Method 1.

$$\frac{P(\text{2B and at least 1B})}{P(\text{at least one B})} = \frac{P(\text{2B})}{1 - P(\text{WW})} = \frac{P(\text{B})P(\text{B})}{1 - P(\text{W})P(\text{W})}$$
$$= \frac{1/4}{1 - 1/4} = \frac{1}{3}$$

Method 2. In the new universe of at least one B, there are 3 equally likely points, BB, BW, WB. One of them is fav, so the prob is 1/3.

(b) This is interpreted as $P(\text{2nd is B}|\text{1st is B})$, which is $P(\text{2nd is B})$, since the balls are painted independently. Answer is 1/2.

11. $P(\text{need more than 3 missiles}) = P(\text{first 3 missiles miss})$
$= P(\text{miss})P(\text{miss})P(\text{miss}) = (.2)^3$

12. **(a)** $P(\text{W on 1st and 4th}) = P(\text{W on 1st and 2nd})$ by symmetry

$$= \frac{10}{15}\frac{9}{14}$$

(b) $\frac{10}{15}\frac{10}{15}$

13. Use the notation HTH for the event "A tosses H, B tosses T, C tosses H."

Method 1. The prob that A is odd man out is the prob of

HTT or THH on round 1

OR

(HHH or TTT on round 1) and (HTT or THH on round 2)

OR

(HHH or TTT on round 1) and (HHH or TTT on round 2) and (HTT or THH on round 3)

OR

⋮

On any one round, events like HHH, HHT, and so on, are mutually exclusive, and the three coin tosses are independent. And the rounds themselves are independent. So

$$P(\text{A is odd man out}) = p_a q_b q_c + q_a p_b p_c \text{ (where } q_a = 1 - p_a, \text{ etc.)}$$
$$+ (p_a p_b p_c + q_a q_b q_c)(p_a q_b q_c + q_a p_b p_c)$$
$$+ (p_a p_b p_c + q_a q_b q_c)^2 (p_a q_b q_c + q_a p_b p_c)$$
$$+ \cdots$$

The series is geometric with $a = p_a q_b q_c + q_a p_b p_c$, $r = p_a p_b p_c + q_a q_b q_c$. So

$$P(\text{A is odd man out}) = \frac{a}{1-r} = \frac{p_a q_b q_c + q_a p_b p_c}{1 - (p_a p_b p_c + q_a q_b q_c)}$$

Method 2. Let p be the prob that A is odd man out. Then

$$p = P(\text{HTT or TTH}) + P(\text{HHH or TTT, and then A is odd man out in the rest of the game})$$
$$= p_a q_b q_c + q_a p_b p_c + (p_a p_b p_c + q_a q_b q_c)p$$

Solve for p to get the same answer as in method 1.

14. Imagine drawing from a deck containing just hearts, spades, clubs.

$$P(\text{heart before black}) = P(\text{heart in 1 draw from the new deck})$$
$$= \frac{13}{39} = \frac{1}{3}$$

15. *Method 1.* Consider a new universe where only 5 and 7 are possible. This cuts down to outcomes (1,4), (4,1), (3,2), (2,3), (3,4), (4,3), (6,1), (1,6), (5,2), (2,5), all equally likely.

$$P(5 \text{ before 7 in repeated tosses}) = P(5 \text{ in restricted universe})$$
$$= \frac{\text{fav}}{\text{total}} = \frac{4}{10}$$

Method 2.

$$P(5) = \frac{4}{36}, P(7) = \frac{6}{36}, P(5 \text{ before } 7) = \frac{4/36}{4/36 + 6/36} = \frac{4}{10}$$

16. **(a)** Consider a new universe with just C_1, C_2.
Then P(C_1 leaves first) = 1/2.

(b) and **(c)** Consider a new universe with just C_1, C_2, C_3.

$$P(C_1 \text{ first}) = \frac{1}{3}$$

$$P(C_1 \text{ then } C_2 \text{ then } C_3) = \frac{\text{fav permutations}}{\text{total perms}} = \frac{1}{3!} = \frac{1}{6}$$

17. $\dfrac{P(1)}{P(1) + P(2)} = \dfrac{2}{5}$

Solutions Section 2-2

1. **(a)** $\dfrac{40!}{30!\,5!\,3!\,2!}(.6)^{30}(.3)^5(.07)^3(.03)^2$

(b) P(30 good, 3 fair, 6 others)= $\dfrac{40!}{30!\,3!\,6!}(.6)^{30}(.3)^4(.1)^6$

(c) $(.97)^{40}$

2. **(a)** $\left(\frac{1}{2}\right)^{16}$ **(b)** $\binom{16}{7}\left(\frac{1}{2}\right)^{16}$

(c) $P(15H) + P(16H) = \binom{16}{15}\left(\frac{1}{2}\right)^{16} + \left(\frac{1}{2}\right)^{16}$

3. $P(3\text{ G} \& 2\text{R or 4G} \& 1\text{R or 5G}) = \binom{5}{3}(.3)^3(.7)^2 + \binom{5}{4}(.3)^4(.7) + (.3)^5$

4. **(a)** $(.3)^5$ **(b)** $\dfrac{5!}{1!\,2!\,2!}(.3)(.6)^2(.1)^2$

(c) The number Against has a binomial distribution where $P(A) = .6$.

$$P(\text{majority Against}) = P(3A) + P(4A) + P(5A)$$
$$= \binom{5}{3}(.6)^3(.4)^2 + \binom{5}{4}(.6)^4(.4) + (.6)^5$$

5. The births are Bernoulli trials where we assume $P(G) = P(B) = 1/2$.

(a) $P(3G) = \binom{6}{3}\left(\frac{1}{2}\right)^6$ **(b)** $\left(\frac{1}{2}\right)^6$ **(c)** $\left(\frac{1}{2}\right)^6$

6. **(a)** This amounts to 6H in 9 tosses. Prob is $\binom{9}{6}(.6)^6(.4)^3$.

(b) $P(\text{at least 9H}|\text{at least 8H})$

$$= \frac{P(\text{at least 9H and at least 8H})}{P(\text{at least 8H})}$$

$$= \frac{P(\text{at least } 9)}{P(\text{at least } 8)} = \frac{P(9) + P(10)}{P(8) + P(9) + P(10)}$$

$$= \frac{\binom{10}{9}(.6)^9(.4) + (.6)^{10}}{\binom{10}{8}(.6)^8(.4)^2 + \binom{10}{9}(.6)^9(.4) + (.6)^{10}}$$

7. **(a)** The tosses are ind trials, each with 5 equally likely outcomes (the boxes).

$$P(2 \text{ of each}) = \frac{10!}{(2!)^5}\left(\frac{1}{5}\right)^{10}$$

(b) Each toss has two outcomes: B_2 with prob 1/5, and elsewhere with prob 4/5.

$$P(10 \text{ elsewheres}) = \left(\frac{4}{5}\right)^{10}$$

(c) Each toss has two outcomes, B_3 and non-B_3.

$$P(6\ B_3\text{'s}) = \binom{10}{6}\left(\frac{1}{5}\right)^6\left(\frac{4}{5}\right)^4$$

(d) P(each box gets at least one ball)

$= 1 - P(B_1 = 0 \text{ or } B_2 = 0 \text{ or } \dots \text{ or } B_5 = 0)$

$$= 1 - \left[\begin{array}{l} P(B_1 = 0) + \cdots + P(B_5 = 0) \left(5 \text{ terms, each is } \left(\frac{4}{5}\right)^{10}\right) \\ -P(B_1 = 0 \,\&\, B_2 = 0) \text{ and other 2-at-a-time terms} \\ \qquad \left(\binom{5}{2} \text{ terms, each is } \left(\frac{3}{5}\right)^{10}\right) \\ + \text{3-at-a-time terms} \\ - \text{4-at-a-time terms} \\ +P(\text{all 5 empty}) \quad \text{(impossible)} \end{array}\right]$$

$$= 1 - \left[5\left(\frac{4}{5}\right)^{10} - \binom{5}{2}\left(\frac{3}{5}\right)^{10} + \binom{5}{3}\left(\frac{2}{5}\right)^{10} - \binom{5}{4}\left(\frac{1}{5}\right)^{10}\right]$$

8. $P(\text{machine fails}) = 1 - P(5 \text{ or } 6 \text{ or } 7 \text{ successful components})$

$$= 1 - \binom{7}{5}(.8)^5(.2)^2 - \binom{7}{6}(.8)^6(.2) - (.8)^7$$

9. **(a)** Each spot in the string is a trial with outcomes 4 versus non-4.

$$P(\text{two 4's}) = \binom{7}{2}(.1)^2(.9)^5$$

(b) Each spot in the string is a trial where the two outcomes are "> 5" with prob .4 and "≤ 5" with prob .6.

$$P(\text{one is} > 5) = \binom{7}{1}(.4)(.6)^6$$

10. $P(\text{pair}) = P(\text{two 6's, one each of } 3, 4, 5)$

$\qquad + P(\text{two 4's, one each of } 2, 3, 6) + \cdots$

There are $6\binom{5}{3}$ terms in the sum (pick a face for the pair and then pick 3 more faces for the singletons). Each prob is

$$\frac{5!}{2!\ 1!\ 1!\ 1!}\left(\frac{1}{6}\right)^5$$

by the multinomial formula. Answer is

$$6\binom{5}{3}\frac{5!}{2!}\left(\frac{1}{6}\right)^5$$

11. **(a)** Draw 10 with replacement from a box of 7G, 14 others.

$$P(4G) = \binom{10}{4}\left(\frac{7}{21}\right)^4\left(\frac{14}{21}\right)^6$$

(b) Draw 10 times with*out* replacement.

$$P(4G) = \frac{\binom{7}{4}\binom{14}{6}}{\binom{21}{10}}$$

12. **(a)** $(.6)^2(.4)^6$

(b) $\binom{8}{2}(.6)^2(.4)^6$

13. The cars are 100 Bernoulli trials with $P(\text{doesn't stop}) = .05$.

$$\begin{aligned} &P(\text{at least 3 no-stops}) \\ &\quad = 1 - P(\text{0 no-stops}) - P(\text{1 no-stop}) - P(\text{2 no-stops}) \\ &\quad = 1 - (.95)^{100} - \binom{100}{1}(.05)(.95)^{99} - \binom{100}{2}(.05)^2(.95)^{98} \end{aligned}$$

14. **(a)** Pick a first box to get 4 balls and then another box to get 2 balls. Then find the prob of getting 4 balls into the first box and 2 into the second.

Answer is

$$10 \cdot 9 \; \frac{6!}{4!\,2!}\left(\frac{1}{10}\right)^6$$

This is really using the OR rule for mutually exclusive events:

$P(\text{4–2 split})$

$= P(\text{4 into } B_1, \text{2 into } B_6) + P(\text{4 into } B_3, \text{2 into } B_1) + \cdots$

There are $10 \cdot 9$ terms, and each term is $\dfrac{6!}{4!\,2!}(1/10)^6$

(b) Pick a pair of boxes. Then find the prob of getting 3 balls in each.

Answer is

$$\binom{10}{2}\frac{6!}{3!\,3!}\left(\frac{1}{10}\right)^6$$

Why use $10 \cdot 9$ in part (a) and $\binom{10}{2}$ in part (b)? Because in (a) there are 2 slots, the box to get the 4 balls and the box to get 2 balls. But in (b), the 2 boxes aren't distinguished from one another since each gets 3 balls; they are just a committee of 2 boxes.

(c) *Method 1.*

$P(\text{all different}) = P(\text{any first})P(\text{second different}|\text{first})\ldots$

$$= 1 \cdot \frac{9}{10}\,\frac{8}{10}\,\frac{7}{10}\,\frac{6}{10}\,\frac{5}{10}$$

Method 2. Pick 6 boxes. Then use the multinomial to find the prob that each of the 6 gets 1 ball. Answer is

$$\binom{10}{6}P(\text{1 each, say, of } B_1, \ldots, B_6 \text{ in 6 trials}) = \binom{10}{6}\frac{6!}{(1!)^6}\left(\frac{1}{10}\right)^6$$

15. A handful means without replacement. So the drawings are not Bernoulli trials and the binomial distribution doesn't apply.

$$P(2L, 2R) = \frac{\binom{10}{2}\binom{12}{2}}{\binom{22}{4}}$$

16. **(a)** $P(\text{first 19 are righties}) = (.85)^{19}$

(b) $P(\text{first 19 are righties and 20th is a lefty}) = (.85)^{19}\,(.15)$

(c) $P(\text{2L in 19 tries and then L on 20th})$

$= P(\text{2L in 19 tries})\,P(\text{L on 20th}) = \binom{19}{2}(.15)^2(.85)^{17}(.15)$

(d) A long way (infinitely many terms) is

$$\begin{aligned} &P(\text{20 tries to get 3L}) + P(\text{21 tries to get 3L}) + \cdots \\ &\quad = \binom{19}{2}(.15)^2(.85)^{17}.15 + \binom{20}{2}(.15)^2(.85)^{18}.15 \\ &\qquad + \binom{21}{2}(.15)^2(.85)^{19}.15 + \cdots \end{aligned}$$

Another fairly long way is

$$\begin{aligned} &1 - P(\text{3 tries to get 3L}) \\ &\quad - P(\text{4 tries to get 3L}) - \cdots - P(\text{19 tries to get 3L}) \\ &= 1 - (.15)^3 - \binom{3}{2}(.15)^2(.85).15 - \cdots - \binom{18}{2}(.15)^2(.85)^{16}.15 \end{aligned}$$

The fastest way is to find

$$\begin{aligned} &P(\text{less than 3L in 19 tries}) \\ &\quad = P(\text{no L or 1L or 2L in 19 tries}) \\ &\quad = (.85)^{19} + \binom{19}{1}(.15)(.85)^{18} + \binom{19}{2}(.15)^2(.85)^{17} \end{aligned}$$

(e) $P(\text{R}^4\text{L or R}^9\text{L or R}^{14}\text{L or} \ldots)$

$$= (.85)^4(.15) + (.85)^9(.15) + (.85)^{14}(.15) + \cdots$$

Geometric series with $a = (.85)^4(.15)$ and $r = (.85)^5$. Answer is

$$\frac{a}{1-r} = \frac{(.85)^4(.15)}{1-(.85)^4}$$

17. $P(\text{T}^9\text{H or H}^9\text{T}) = q^9p + p^9q.$

Solutions Section 2-4

1. $P(\text{CB}) = P(\text{M})P(\text{CB}|\text{M}) + P(\text{W})P(\text{CB}|\text{W}) = \#1 + \#3$

$$= (.53)(.02) + (.47)(.001)$$

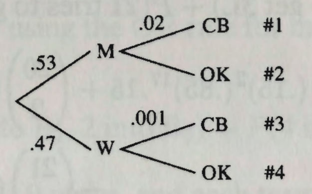


Figure P.1

2. $P(\text{2nd is A}_\text{S})$

$$= P(\text{1st is spade})\ P(\text{2nd is A}_\text{S}|\text{1st is spade})$$

$$+P(\text{1st is non-S})P(\text{2nd is A}_\text{S}|\text{1st is non-S})$$

$$= \#1 + \#3 = \frac{13}{52}\frac{1}{52} + \frac{39}{52}\frac{1}{51}$$

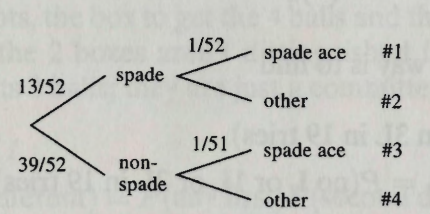


Figure P.2

3. (a) $P(\text{right}) = \#1 + \#3 = (.75)(.8) + (.25)(.2) = .65$

(expect 65% on the exam)

(b) $P(\text{guess}|\text{right}) = \dfrac{\#3}{\#1 + \#3} = \dfrac{(.25)(.2)}{65} \approx .07$

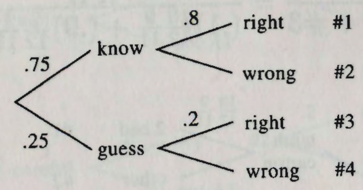


Figure P.3

4. $P(\text{B}|\text{W}) = \dfrac{P(\text{B and W})}{P(\text{W})} = \dfrac{\#3}{\#1 + \#3 + \#5} = \dfrac{\frac{1}{3}\frac{7}{15}}{\frac{1}{3}\frac{1}{3} + \frac{1}{3}\frac{7}{15} + \frac{1}{3}\frac{4}{9}}$

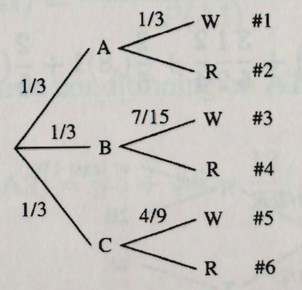


Figure P.4

5. $P(\text{at least 4H}) = \#1 + \#3$

$$= P(\text{H on 1st})P(\text{3H or 4H or 5H later}|\text{H on 1st})$$

$$+P(\text{T on 1st})P(\text{4H or 5H or 6H later}|\text{T on 1st})$$

$$= \frac{2}{3}\left[\binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + \binom{5}{4}\left(\frac{2}{3}\right)^4\frac{1}{3} + \left(\frac{2}{3}\right)^5\right]$$

$$+\frac{1}{3}\left[\binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + \binom{6}{5}\left(\frac{2}{3}\right)^5\frac{1}{3} + \left(\frac{2}{3}\right)^6\right]$$

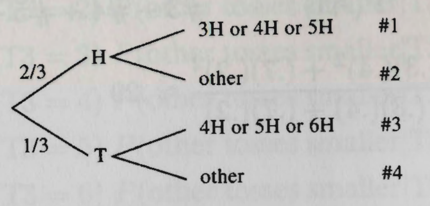


Figure P.5

6. $$P(\text{tenth}|2B) = \frac{P(\text{tenth and }2B)}{P(2\text{ bad})}$$

$$= \frac{\#1}{\#1 + \#3} = \frac{(.1)\frac{10}{12}\frac{9}{11}}{(.1)\frac{10}{12}\frac{9}{11} + (.9)\frac{2}{12}\frac{1}{11}}$$

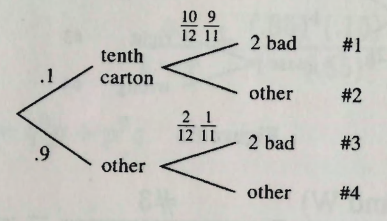


Figure P.6

7. $P(\text{at least one W on 2nd draw}) = \#1 + \#3 + \#5 + \#7$

$$= \frac{3}{5}\frac{1}{2}\left(1 - \frac{2}{4}\frac{1}{3}\right) + \frac{3}{5}\frac{1}{2}\frac{2}{4} + \frac{2}{5}(.8)1 + \frac{2}{5}(.2)\frac{3}{4}$$

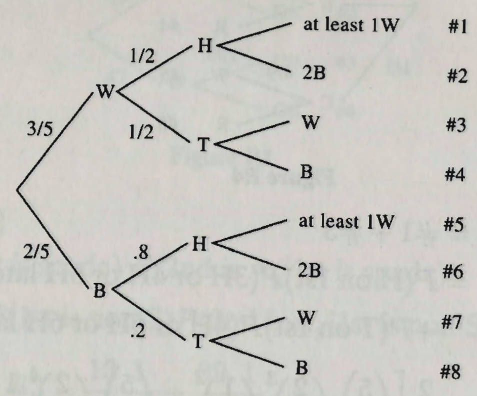


Figure P.7

8. *Method 1.* Let A2 denote accident next year and let A1 denote accident this year.

$$P(A2|A1) = \frac{P(A2\text{ and }A1)}{P(A1)} = \frac{\#3 + \#5}{\#3 + \#4 + \#5 + \#6} = \frac{\#3 + \#5}{\#1 + \#2}$$

$$= \frac{(.3)(.4)^2 + (.7)(.2)^2}{(.3)(.4) + (.7)(.2)} \approx .29$$

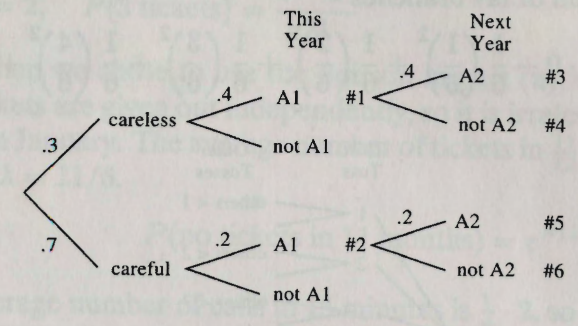


Figure P.8 Method 1

Method 2. First find the prob of careless given A1.

$$P(\text{careless}|A1) = \frac{P(\text{careless and }A1)}{P(A1)} = \frac{\#1}{\#1 + \#3} = \frac{12}{26}$$

Then make a new tree conditioning on A1. In this conditional world,

$$P(A2) = \#5 + \#6 = \frac{12}{26}(.4) + \frac{14}{26}(.2)$$

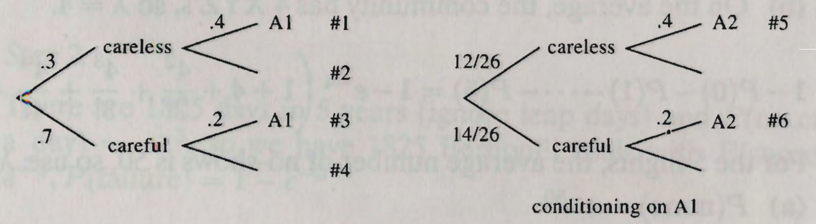


Figure P.8 Method 2

9. (a) $P(A)$ (b) $P(C|A)$ (c) $P(A\text{ and }C)$

10. $P(\text{first 2 tosses are less than third})$

$$= P(T3 = 1)\ P(\text{other tosses smaller}|T3 = 1)$$
$$+P(T3 = 2)\ P(\text{other tosses smaller}|T3 = 2)$$
$$+P(T3 = 3)\ P(\text{other tosses smaller}|T3 = 3)$$
$$+P(T3 = 4)\ P(\text{other tosses smaller}|T3 = 4)$$
$$+P(T3 = 5)\ P(\text{other tosses smaller}|T3 = 5)$$
$$+P(T3 = 6)\ P(\text{other tosses smaller}|T3 = 6)$$

= sum of fav branches

$$= \frac{1}{6}\cdot 0 + \frac{1}{6}\left(\frac{1}{6}\right)^2 + \frac{1}{6}\left(\frac{2}{6}\right)^2 + \frac{1}{6}\left(\frac{3}{6}\right)^2 + \frac{1}{6}\left(\frac{4}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{6}\right)^2$$

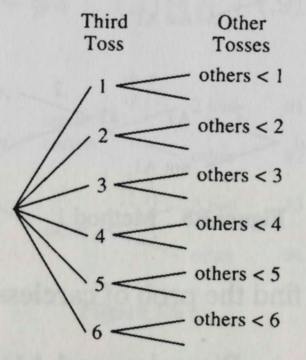


Figure P.10

Solutions Section 2-5

1. **(a)** Use the Poisson with $\lambda = 2$.

$$1 - P(0) - P(1) - P(2) = 1 - e^{-2} - 2e^{-2} - \frac{4e^{-2}}{2!}$$

(b) On the average, the community has 4 XYZ's, so $\lambda = 4$.

$$1 - P(0) - P(1) - \cdots - P(5) = 1 - e^{-4}\left(1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right)$$

2. For the 5 flights, the average number of no-shows is 50, so use $\lambda = 50$.

(a) $P(\text{none}) = e^{-50}$

(b) $P(4) = \dfrac{50^4 e^{-50}}{4!}$

(c) $P(\text{none}) + P(1) + P(2) + P(3) + P(4)$

$$= e^{-50}\left(1 + 50 + \frac{50^2}{2!} + \frac{50^3}{3!} + \frac{50^4}{4!}\right)$$

3. **(a)** Use the binomial with $n = 100$, $p = .05 = P(\text{fail to stop})$.

$$P(\text{at least 2 fail}) = 1 - P(\text{none}) - P(1)$$
$$= 1 - (.95)^{100} - 100(.05)(.95)^{99}$$

(b) $\lambda = 3$, $P(\text{at least 2}) = 1 - P(\text{none}) - P(1) = 1 - e^{-3} - 3e^{-3}$

4. **(a)** $\lambda = 2$, $\quad P(3\text{ tickets}) = \dfrac{e^{-2}\,2^3}{3!}$

(b) When we chose to use the Poisson in part (a), we assumed that tickets are given out independently, so it is irrelevant that you got 2 in January. The average number of tickets in $\frac{11}{12}$ of a year is $\frac{11}{12}\cdot 2$, so $\lambda = 11/6$.

$$P(\text{no tickets in 11 months}) = e^{-11/6}$$

5. The average number of calls in 15 minutes is $\frac{1}{4}\cdot 2$, so $\lambda = 1/2$.

(a) $P(\text{no calls in 15 minutes}) = e^{-1/2}$.

(b) $P(\text{no more than 1 call in 15 minutes})$
$= P(\text{none}) + P(1) = e^{-1/2} + \frac{1}{2}e^{-1/2}$

6. On the average there are $\lambda_1 + \lambda_2 + \lambda_3$ disasters in a year, so $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

$$P(\text{at least 1 disaster}) = 1 - P(\text{none}) = 1 - e^{-(\lambda_1+\lambda_2+\lambda_3)}$$

7. *Step 1.*

The number of calls in a day is a Poisson random variable with $\lambda = 3$.

$P(\text{no calls in a day}) = e^{-3}$

Step 2.

There are 1825 days in 5 years (ignore leap days) and $P(\text{no calls in a day}) = e^{-3}$, so we have 1825 Bernoulli trials with $P(\text{success}) = e^{-3}$, $P(\text{failure}) = 1 - e^{-3}$.

$$P(\text{at least 1 success}) = 1 - P(\text{no successes}) = 1 - (1 - e^{-3})^{1825}$$

8. $\lambda = np = (1000)(.01) = 10$, Poisson approximation is $10e^{-10}$.

9. Bernoulli trials are independent repetitions of the same experiment where the experiment has two outcomes, success and failure (coin tosses).

Solutions Review Problems for Chapters 1, 2

1. **(a)** *With Replacement (multinomial)*

$$P(3\text{W}, 4\text{R}, 3\text{ others}) = \frac{10!}{3!\,4!\,3!}\left(\frac{20}{140}\right)^3\left(\frac{40}{140}\right)^4\left(\frac{80}{140}\right)^3$$

W/O Replacement

$$P(\text{3W, 4R, 3 others}) = \frac{\text{fav}}{\text{total}} = \frac{\binom{20}{3}\binom{40}{4}\binom{80}{3}}{\binom{140}{10}}$$

(b) *With*

$$P(\text{WWW RRRR OOO}) = \left(\frac{20}{140}\right)^3\left(\frac{40}{140}\right)^4\left(\frac{80}{140}\right)^3$$

W/O

$$\frac{20}{140}\,\frac{19}{139}\,\frac{18}{138}\,\frac{40}{137}\,\frac{39}{136}\,\frac{38}{135}\,\frac{37}{134}\,\frac{80}{133}\,\frac{79}{132}\,\frac{78}{131}$$

(c) Same as part (b) by symmetry

2. Nine Bernoulli trials where on any one trial $P(2) = .1$.

$P(\text{at least four 2's})$

$$= 1 - P(\text{no 2's}) - P(\text{one 2}) - P(\text{two 2's}) - P(\text{three 2's})$$

$$= 1 - (.9)^9 - \binom{9}{1}(.1)(.9)^8 - \binom{9}{2}(.1)^2(.9)^7 - \binom{9}{3}(.1)^3(.9)^6$$

3. (a) By symmetry,

$P(\text{10th is king, 11th is non-K})$

$$= P(\text{K on 1st, non-K on 2nd})$$

$$= \frac{4}{52}\,\frac{48}{51}$$

(b) *Without Replacement*

$$P(\text{K}^9\text{K}) = \frac{48}{52}\,\frac{47}{51}\,\frac{46}{50}\cdots\frac{40}{44}\,\frac{4}{43}$$

With Replacement

$$\left(\frac{48}{52}\right)^9\frac{1}{13}$$

(c) $P(\text{2K in 9 draws, then K on 10th})$

$$= P(\text{2K in 9 draws})P(\text{K on 10th}|\text{2K in first 9 draws}) = \frac{\binom{4}{2}\binom{48}{7}}{\binom{52}{9}}\,\frac{2}{43}$$

(d) $P(\text{no K or 1K or 2K in first 9 draws})$

$$= P(\text{no K}) + P(\text{1K}) + P(\text{2K})$$

$$= \frac{\binom{48}{9} + 4\binom{48}{8} + \binom{4}{2}\binom{48}{7}}{\binom{52}{9}}$$

4. $P(\text{at least one card } < 6 | \text{at least one card } > 9)$

$$= \frac{P(\text{at least one } < 6 \text{ and at least one } > 9)}{P(\text{at least one } > 9)}$$

$$\text{denominator} = 1 - P(\text{all } \le 9) = 1 - \frac{\binom{32}{13}}{\binom{52}{13}}$$

$$\text{numerator} = 1 - P(\text{all cards } \ge 6 \text{ or all } \le 9)$$

$$= 1 - [P(\text{all } \ge 6) + P(\text{all } \le 9) - P(6 \le \text{ all } \le 9)]$$

$$= 1 - \frac{\binom{36}{13} + \binom{32}{13} - \binom{16}{13}}{\binom{52}{13}}$$

5. (a) The symbols in the string are Bernoulli trials where each trial results in vowel or non-vowel.

$$P(\text{3 vowels}) = \binom{12}{3}\left(\frac{5}{36}\right)^3\left(\frac{31}{36}\right)^9$$

(b) $P(\text{3 vowels}) = \dfrac{\text{fav}}{\text{total}} = \dfrac{\binom{5}{3}\binom{31}{9}}{\binom{36}{12}}$

6. This is like drawing 8 balls without replacement from a box containing 2L, 3I, 1N, 1S.

Method 1.

$P(\text{1st and last are L's})$

$$= P(\text{1st and 2nd are L's}) \text{ (by symmetry)}$$

$$= P(\text{1st is L}) + P(\text{2nd is L}) - P(\text{1st and 2nd are L's})$$

$$= 2P(\text{1st is L}) - P(\text{1st and 2nd are L's}) \text{ (more symmetry)}$$

$$= 2 \cdot \frac{2}{8} - \frac{2}{8}\,\frac{1}{7}$$

Method 2.

$$P(\text{1st and last are L's}) = P(\text{1st and 2nd are L's}) \text{ (by symmetry)}$$
$$= 1 - P(\text{1st and 2nd are non-L's}) = 1 - \frac{6}{8}\frac{5}{7}$$

7. **(a)** Here's one way to do it. The total number of ways to pick 2 of the 8 seats for J and M is $\binom{8}{2}$. There are 7 fav ways (seats 1 and 2, seats 2 and 3, ..., seats 7 and 8). Answer is $7/\binom{8}{2}$.

(b) A circular table is tricky. For example, the circle *ABCDEFGH* is the same as the circle *BCDEFGHA*. Here's one method. Put John down anywhere. When Mary sits down there are 7 seats available and 2 are fav. So prob = 2/7.

8. $P(\text{word contains } z) = 1 - P(\text{no } z\text{'s}) = 1 - \left(\frac{25}{26}\right)^3$.

9. There are 365 Bernoulli trials, and on any one trial $P(\text{towed}) = .1$.

$$P(\text{at most one tow}) = P(\text{none}) + P(\text{one})$$
$$= (.9)^{365} + \binom{365}{1}(.9)^{364}(.1)$$

10. This is drawing balls from a box without replacement. By symmetry,

$$P(\text{last two are M}) = P(\text{first two are M}) = \frac{m}{m+w}\cdot\frac{m-1}{m+w-1}$$

11. **(a)** There are j married men and k single men, so the man can be picked in $j+k$ ways. Similarly, the woman can be picked in $j+n$ ways. Total number of ways of picking the pair is $(j+k)(j+n)$. There are $j \cdot j$ fav ways, so

$$P(\text{both married}) = \frac{j^2}{(j+k)(j+n)}$$

(b) $P(\text{man married, woman single}) + P(\text{man single, woman married})$

$$= \frac{jn+kj}{(j+k)(j+n)}$$

(c) There are j favs (the man and the woman have to be one of the j married couples).

$$\frac{j}{(j+k)(j+n)}$$

12. **(a)** $P(\text{non-3 in one toss}) = 5/6$, so $P(\text{non-3 in 10 tosses}) = (5/6)^{10}$

(b) $(5/6)^{100000}$

(c) $\lim_{n\to\infty}(5/6)^n = (5/6)^\infty = 0$

13. $P(\text{match}) = P(\text{2 black or 2 blue or 2 white})$
$= P(\text{2 black}) + P(\text{2 blue}) + P(\text{2 white})$

Method 1. Treat the pair of socks as a committee.

$$P(\text{match}) = \frac{\binom{5}{2}+\binom{6}{2}+\binom{7}{2}}{\binom{18}{2}}$$

Method 2. We'll get the same answer if we let order count (as long as we do it consistently in the numerator and denominator).

$$P(\text{match}) = \frac{5}{18}\frac{4}{17} + \frac{6}{18}\frac{5}{17} + \frac{7}{18}\frac{6}{17}$$

14. *Method 1.*

$$P(\text{H on 8th}|\text{6H,4T}) = \frac{P(\text{6H, 4T and H on 8th})}{P(\text{6H and 4T})}$$

$$\text{denom} = \binom{10}{6}\left(\frac{1}{2}\right)^{10} \quad \text{(binomial distribution)}$$

$$\text{numerator} = P(\text{H on 8th})P(\text{5H,4T in 9 throws}) \quad \text{(by independence)}$$
$$= \frac{1}{2}\binom{9}{5}\left(\frac{1}{2}\right)^9$$

Method 2. Think of an urn containing 6H and 4T. Draw w/o replacement.

$$P(\text{8th is H}) = P(\text{1st is H}) = \frac{6}{10}$$

15. **(a)** (multinomial)

$$P(\text{1A, 1B, 4 others in 6 trials}) = \frac{6!}{1!\,1!\,4!}\frac{1}{26}\frac{1}{26}\left(\frac{24}{26}\right)^4$$

(b) $1 - P(\text{no A or no B})$

$$= 1 - [P(\text{no A}) + P(\text{no B}) - P(\text{no A and no B})]$$
$$= 1 - \left[2\left(\frac{25}{26}\right)^6 - \left(\frac{24}{26}\right)^6\right]$$

(c) $1 - P(\text{no A and no B}) = 1 - (24/26)^6$

(d) $P(\text{2A's}) - P(\text{2A and no B}) - P(\text{2A and 1B})$

$$= \binom{6}{2}\left(\frac{1}{26}\right)^2\left(\frac{25}{26}\right)^4 - \frac{6!}{2!\,0!\,4!}\left(\frac{1}{26}\right)^2\left(\frac{24}{26}\right)^4$$

$$- \frac{6!}{2!\,1!\,3!}\left(\frac{1}{26}\right)^3\left(\frac{24}{26}\right)^3$$

(e) *Method 1.*

$$P(\text{any first})P(\text{different}|\text{first})\cdots = 1 \cdot \frac{25}{26}\frac{24}{26}\frac{23}{26}\frac{22}{26}\frac{21}{26}$$

Method 2. Pick 6 flavors. Then find the prob of getting one each of those 6.

$$\binom{26}{6} P(\text{one each of, say, A,B,C,D,E,F}) = \binom{26}{6}\frac{6!}{(1!)^6}\left(\frac{1}{26}\right)^6$$

(f) *Method 1.*

$$P(\text{any first})P(\text{same}|\text{first})\cdots = 1\left(\frac{1}{26}\right)^5$$

Method 2.

$$P(\text{all A or all B or} \ldots \text{or all Z})$$

$$= P(\text{all A}) + \cdots + P(\text{all Z}) = 26\left(\frac{1}{26}\right)^6$$

16. (a) The numerator double counts. For instance, it counts the following outcomes as different when they are the same:

outcome 1 Pick spades to be the missing suit.
Pick all the hearts as your 13 non-spades.

outcome 2 Pick diamonds to be the missing suit.
Pick all the hearts as your 13 non-diamonds.

(b) $P(\text{at least one suit missing})$

$= P(\text{no H or no S or no D or no C})$

$= P(\text{no H}) + P(\text{no S}) + P(\text{no D}) + P(\text{no C})$

$- P(\text{no H and no S})$ and other 2-at-a-time terms

$\left(\binom{4}{2}\right.$ terms, each is $\left.\binom{26}{13}\Big/\binom{52}{13}\right)$

$+ P(\text{no H, no S, no D})$ and other 3-at-a-time terms

$- P(\text{no H, no S, no C, no D})$ (impossible event)

$$= \frac{4\binom{39}{13} - \binom{4}{2}\binom{26}{13} + \binom{4}{3}\cdot 1}{\binom{52}{13}}$$

17. (a) Consider a new box containing only B_3 and B_5.

$P(B_3 \text{ before } B_5 \text{ from old box})$

$= P(B_3 \text{ in one draw from new box})$

$= \frac{1}{2}$

(b) Consider a new box containing only B_3 and the 5 whites.

$P(B_3 \text{ before white from old box})$

$= P(B_3 \text{ in one draw from new box}) = \frac{1}{6}$

18. $$P(\text{Z}|\text{wrong}) = \frac{P(\text{Z and wrong})}{P(\text{wrong})} = \frac{\#3}{\#1 + \#2 + \#3}$$

$$= \frac{(.1)(.04)}{(.6)(.02) + (.3)(.03) + (.1)(.04)}$$

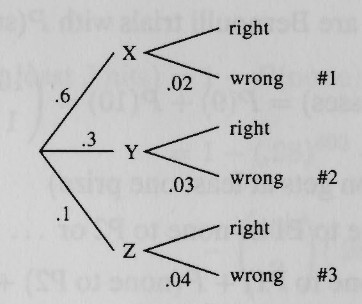


Figure P.18

19. $$P(\text{at most 5H}|\text{at least 3H}) = \frac{P(\text{at most 5 and at least 3})}{P(\text{at least 3})}$$

$$= \frac{P(\text{3 or 4 or 5})}{1 - P(\text{0 or 1 or 2})}$$

$$= \frac{\binom{10}{3}\left(\frac{1}{2}\right)^{10} + \binom{10}{4}\left(\frac{1}{2}\right)^{10} + \binom{10}{5}\left(\frac{1}{2}\right)^{10}}{1 - \left(\frac{1}{2}\right)^{10} - \binom{10}{1} - \left(\frac{1}{2}\right)^{10} - \binom{10}{2}\left(\frac{1}{2}\right)^{10}}$$

20. Let X and Y be the number of throws needed, respectively, by the two players.

$$P(X=Y) = P(X=1, Y=1) + P(X=2, Y=2) + \cdots$$
$$= P(X=1)P(Y=1) + P(X=2)P(Y=2) + \cdots \quad \text{(independence)}$$
$$= [P(X=1)]^2 + [P(X=2)]^2 + [P(X=3)]^2 + \cdots$$

Here's how to calculate these probs.

The prob of a lucky throw on any one toss is 8/36 = 2/9. So

$$P(X=100) = P(\text{need 100 throws to get lucky})$$
$$= P(\text{99 unluckies followed by a lucky}) = \left(\frac{7}{9}\right)^{99} \frac{2}{9}$$

and

$$P(X=Y) = \left(\frac{2}{9}\right)^2 + \left[\frac{7}{9}\frac{2}{9}\right]^2 + \left[\left(\frac{7}{9}\right)^2 \frac{2}{9}\right]^2 + \left[\left(\frac{7}{9}\right)^3 \frac{2}{9}\right]^2 + \cdots$$

This is a geometric series with $a = (2/9)^2, r = (7/9)^2$. Answer is

$$\frac{4/81}{1 - 49/81} = \frac{1}{8}$$

21. The 10 foul shots are Bernoulli trials with $P(\text{success}) = .85$.

$$P(\text{at least 9 successes}) = P(9) + P(10) = \binom{10}{1}(.85)^9(.15) + (.85)^{10}$$

22. (a) $P(\text{each person gets at least one prize})$

$$= 1 - P(\text{none to P1 or none to P2 or} \ldots \text{or none to P5})$$

$$= 1 - \begin{bmatrix} P(\text{none to P1}) + P(\text{none to P2}) + \cdots \\ -[P(\text{none to P1,P2}) + \cdots] \\ +[P(\text{none to P1,P2,P3}) + \cdots] \\ -[P(\text{none to P1,P2,P3,P4}) + \cdots] \\ +P(\text{none to P1,P2,P3,P4,P5}) \end{bmatrix}$$

$$= 1 - \left[5\left(\frac{4}{5}\right)^{10} - \binom{5}{2}\left(\frac{3}{5}\right)^{10} + \binom{5}{3}\left(\frac{2}{5}\right)^{10} - \binom{5}{4}\left(\frac{1}{5}\right)^{10} + 0\right]$$

(b) $P(\text{no repeats}) = \dfrac{\text{fav}}{\text{total}} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{10^5}$

(c) $P(5\text{M}) = \left(\dfrac{6}{10}\right)^5$

23. *Method 1.*

$$P(\text{11 penny heads}|\text{17 total heads}) = \frac{P(\text{11 penny H and 17 total H})}{P(\text{17 total H})}$$
$$= \frac{P(\text{11 penny H and 6 nickel H})}{P(\text{17H})}$$

$$\text{denominator} = \binom{40}{17}(.7)^{17}(.3)^{23}$$

$$\text{numerator} = \binom{20}{11}(.7)^{11}(.3)^9\binom{20}{6}(.7)^6(.3)^{14}$$

Method 2. Think of a box with 17H and 23T. Draw 20 without replacement.

$$P(\text{11H in 20 draws}) = \frac{\binom{17}{11}\binom{23}{9}}{\binom{40}{20}}$$

24. (a) There are 400 Bernoulli trials with $P(\text{hits your block}) = 1/50 = .02$.

$$P(\text{at least 3 hits}) = 1 - P(\text{none}) - P(1) - P(2)$$
$$= 1 - (.98)^{400} - \binom{400}{1}(.98)^{399}(.02) - \binom{400}{2}(.98)^{398}(.02)^2$$

(b) Use $\lambda = np = 8$. Prob is approximately $1 - e^{-8} - \dfrac{64e^{-8}}{2!}$.

25. Let's call it a success on a round if there is no odd man out. On any round,

$$P(\text{success}) = P(\text{HHH or TTT}) = p^3 + q^3$$
$$P(\text{game lasts at least 6 rounds}) = P(S^5) = (p^3 + q^3)^5$$

26. **(a)** $P(A) + P(B) + P(C) = .8$.

(b) *Method 1.*

$$P(A) + P(B) + P(C) - [P(AB) + P(AC) + P(BC] + P(ABC)$$
$$= .5 + .2 + .1 - [(.5)(.2) + (.5)(.1) + (.2)(.1)] + (.5)(.2)(.1) = .64$$

Method 2.

$$1 - P(\bar{A} \text{ and } \bar{B} \text{ and } \bar{C}) = 1 - (.5)(.8)(.9)$$

27. **(a)** Assuming 4 weeks to a month, on the average, there are 1/16 failures per week. Use the Poisson with $\lambda = 1/16$.

$P(\text{at least one failure during exam week})$
$= 1 - P(\text{no failures})$
$= 1 - e^{-1/16}$

(b) Use $\lambda = 1/4$. $P(\text{no failures in the next month}) = e^{-1/4}$.

28. The people are ind trials where each trial has 12 equally likely outcomes.

$P(\text{3 at one stop, 2 at another}) = P(3S_6\text{'s}, 2S_5\text{'s} + P(3S_1\text{'s}, 2S_7\text{'s}+) \cdots$

There are $12 \cdot 11$ terms in the sum (pick a stop for the trio, pick a stop for the pair). Each prob is

$$\frac{5!}{2!\,3!}\left(\frac{1}{12}\right)^5 \text{ (multinomial formula).}$$

Answer is

$$12 \cdot 11 \frac{5!}{2!\,3!}\left(\frac{1}{12}\right)^5$$

29. **(a)** 999 **(b)** 1 **(c)** 1000

(d) $\dfrac{(n+m-1)!}{(n-1)!\,m!}\dfrac{n!\,m!}{(n+m)!} = \dfrac{n}{n+m}$

30. $P(\text{ends in 6 games})$
$= P(\text{A wins in 6 games})$
$= P(\text{B wins in 6 games})$

$= 2P(\text{A wins in 6 games})$
$= 2P(\text{3A's and 2B's in first 5 games})P(\text{A wins 6th})$
$= 2\binom{5}{3}\left(\frac{1}{2}\right)^5 \frac{1}{2}$

31. The side of the square is $R\sqrt{2}$. Its area is $2R^2$.

$$P(\text{shot lands in square}) = \frac{\text{fav area}}{\text{total area}} = \frac{2R^2}{\pi R^2} = \frac{2}{\pi}$$
$$P(\text{shot lands in I}) + P(\text{II}) + P(\text{III}) + P(\text{IV}) = 1 - \frac{2}{\pi} = \frac{\pi - 2}{\pi}$$
$$P(\text{I}) = P(\text{II}) = P(\text{III}) = P(\text{IV}) = \frac{1}{4}\frac{\pi - 2}{\pi}$$

The 5 shots are independent trials.

(a) $P(\text{all same zone})$
$= P(\text{5 I or 5 II or 5 III or 5 IV or 5 V})$
$= P(\text{5I}) + P(\text{5II}) + P(\text{5III}) + P(\text{5IV}) + P(\text{5V})$
$= 4\left(\frac{1}{4}\frac{\pi-2}{\pi}\right)^5 + \left(\frac{2}{\pi}\right)^5$

(b) $P(\text{all different zones}) = P(\text{1 each of I,II,III,IV,V})$
$= \frac{5!}{(1!)^5}\left(\frac{1}{4}\frac{\pi-2}{\pi}\right)^4 \frac{2}{\pi}$

32. Let M stand for Mother Has XYZ.

Let C_1 stand for First Child Has XYZ, and so on.

Method 1.

$$P(\bar{C}_4|\bar{C}_1\bar{C}_2\bar{C}_3) = \frac{\#1 + \#3}{\#1 + \#2 + \#3 + \#4} = \frac{\#1 + \#3}{\#5 + \#6}$$
$$= \frac{1/32 + 1/2}{1/16 + 1/2} = \frac{17}{18}$$

Method 2. First, find the prob of $\bar{M}$ given $\bar{C}_1\bar{C}_2\bar{C}_3$.

$$P(\bar{M}|\bar{C}_1\bar{C}_2\bar{C}_3) = \frac{\#8}{\#7 + \#8} = \frac{1/2}{1/16 + 1/2} = \frac{8}{9}$$

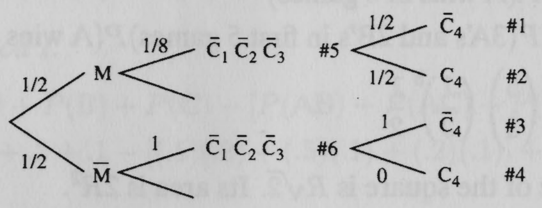


Figure P.32a Method 1

Now make a new tree conditioning on $\bar{C}_1\bar{C}_2\bar{C}_3$. In this conditional world,

$$P(\bar{C}_4) = \#9 + \#10 = \frac{1}{18} + \frac{8}{9} = \frac{17}{18}$$

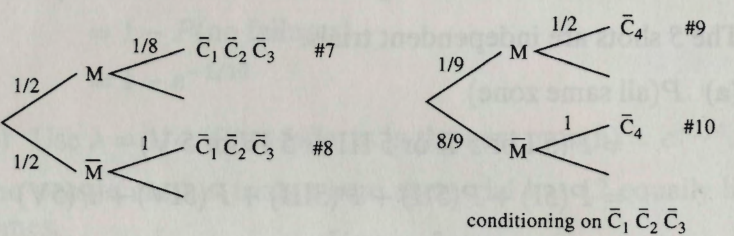


Figure P.32b Method 2

33. $P(\text{at least 2 with the same birthday})$

$$= 1 - P(\text{all different birthdays})$$

$$= 1 - \frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot (365 - n + 1)}{365^n}$$

It turns out that if n is as small as 23, this prob is $\geq .5$. And if $n = 50$, the prob is .970. So it is more likely than you might think for people to share a birthday.

34. $P(6 \text{ from one die in your 3 chances}) = 1 - P(\text{no 6's in 3 tosses})$

$$= 1 - (5/6)^3 = 91/216.$$

Now each of the 5 dice that can be tossed as many as 3 times each is a Bernoulli trial, where

$$P(\text{success}) = P(6 \text{ from your 3 chances}) = \frac{91}{216}$$

$$P(2 \text{ successes in 5 trials}) = \binom{5}{2}\left(\frac{91}{216}\right)^2\left(\frac{125}{216}\right)^3$$

35. By Bayes' theorem,

$$P(xxxxx \text{ sent}|2x, 3y \text{ received}) = \frac{\#1}{\#1 + \#2} = \frac{.6p_1}{.6p_1 + .4p_2}$$

Now we need p_1 and p_2. Each of the five symbols sent is a Bernoulli trial; the outcome is either error or no-error where $P(\text{error}) = .1$. So

$$p_1 = P(3 \text{ errors}) = \binom{5}{3}(.1)^3(.9)^2, \quad p_2 = P(2 \text{ errors}) = \binom{5}{2}(.1)^2(.9)^3$$

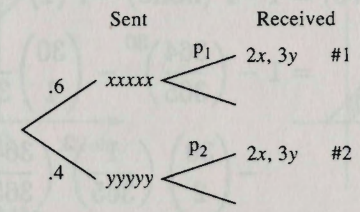


Figure P.35

36. Let x be her score; let y be his score.

(a) $P(y \geq 2x) = \frac{\text{fav area}}{\text{total}} = \frac{3 - \text{unfav}}{3} = \frac{3-1}{3} = \frac{2}{3}$

(b) $P\left(\max \leq \frac{1}{2}\right) = \frac{\text{fav}}{\text{total}} = \frac{1/4}{3} = \frac{1}{12}$

(c) $P\left(\min \geq \frac{1}{2}\right) = \frac{\text{fav}}{\text{total}} = \frac{5/4}{3} = \frac{3}{12}$

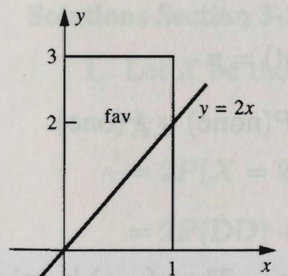


Figure P.36a

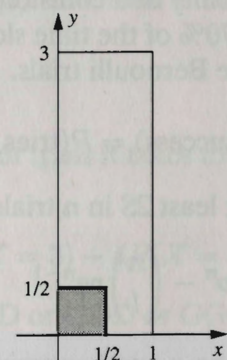


Figure P.36b

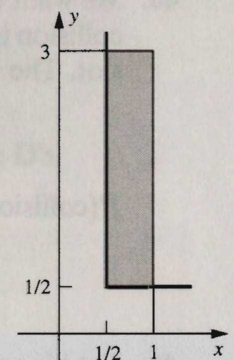


Figure P.36c

37. **(a)** $\binom{19}{4} \Big/ \binom{20}{5}$

(b) $P(\text{J on 1st draw}) + P(\text{J on 2nd draw}) + \cdots + P(\text{J on 5th draw})$

$$= 5P(\text{J on 1st draw}) \quad \text{(by symmetry)} = \frac{5}{20}$$

(c) $1 - \frac{\binom{19}{5}}{\binom{20}{5}}$

38. Each person is a Bernoulli trial where $P(\text{July 4}) = 1/365$.

$$P(\text{at least 3 July 4's}) = 1 - P(\text{none}) - P(1) - P(2)$$

$$= 1 - \left(\frac{364}{365}\right)^{30} - \binom{30}{1}\frac{1}{365}\left(\frac{364}{365}\right)^{29} - \binom{30}{2}\left(\frac{1}{365}\right)^{2}\left(\frac{364}{365}\right)^{28}$$

39. **(a)** Each freshman is a Bernoulli trial where $P(\text{success}) = P(\text{A}) = .2$.

$$P(\text{4A}) = \binom{10}{4}(.2)^4(.8)^6$$

(b) Draw 6 times without replacement (no one can get two offices) from a population of 10F, 20S, 30J, 20G.

$$P(\text{4F}) = \frac{\binom{10}{4}\binom{70}{2}}{\binom{80}{6}}$$

40. We want the probability of a collision in a time slot. (If the prob of a collision is .7, then 70% of the time slots are wasted.) Consider a time slot. The n hosts are Bernoulli trials.

$$P(\text{success}) = P(\text{tries to use slot}) = p$$

$$P(\text{collision}) = P(\text{at least 2S in } n \text{ trials}) = 1 - P(\text{none}) - P(\text{one})$$

$$= 1 - p^n - \binom{n}{1}pq^{n-1}$$

41. Let his arrival time be x and her arrival time be y. Then (x, y) is uniformly distributed in a rectangle.

(a) $P(\text{meet}) = P(|y - x| \le 10) = \dfrac{\text{fav area}}{\text{total}} = \dfrac{1150}{5400}$

(b) $P(\text{J arrives first and then Mary arrives no more than}$

$$\text{10 minutes later}) = P(x \le y \le x + 10) = \frac{\text{fav area}}{\text{total}} = \frac{600}{5400}$$

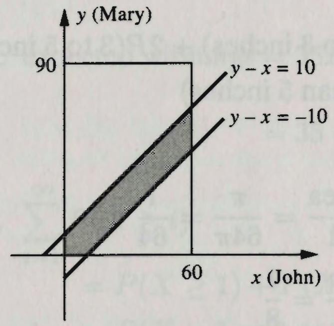


Figure P.41a

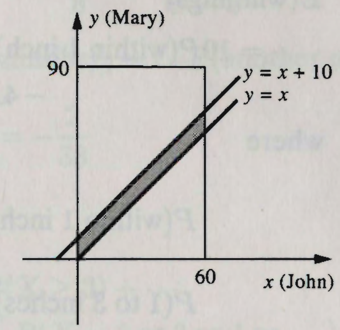


Figure P.41b

42. **(a)** Total is $(100)^{20}$ (each of the 20 spots can be filled in 200 ways). For the fav, we need the number of ways of lining up 20 out of the 100 numbers in increasing order. First, pick 20 (different) numbers out of the 100 to be in the lineup. Then there is only one way to line them up (namely, in increasing order). So the number of fav is $\binom{100}{20}$. Answer is

$$\frac{\binom{100}{20}}{(100)^{20}}$$

(b) Same fav as part (a). The total is $100 \cdot 99 \cdot 98 \cdot \cdots \cdot 81$.

Solutions Section 3-1

1. Let X be the number of trials it takes to locate the D's.

$$\begin{aligned} &EX \\ &= 2P(X = 2) + 3P(X = 3) + 4P(X = 4) \\ &= 2P(\text{DD}) + 3P(\text{DGD or GDD or GGG}) \\ &\quad + 4P(\text{DGG or GDG or GGD}) \\ &= 2\left(\frac{2}{5}\frac{1}{4}\right) + 3\left(\frac{2}{5}\frac{3}{4}\frac{1}{3} + \frac{3}{5}\frac{2}{4}\frac{1}{3} + \frac{3}{5}\frac{2}{4}\frac{1}{3}\right) + 4\left(\frac{2}{5}\frac{3}{4}\frac{2}{3} + \frac{3}{5}\frac{2}{4}\frac{2}{3} + \frac{3}{5}\frac{2}{4}\frac{2}{3}\right) \end{aligned}$$