Sets and Probability

In a survey of 200 people that had just returned from a trip to Europe, the following information was gathered.

- 142 visited England
- 95 visited Italy
- 65 visited Germany
- 70 visited both England and Italy
- 50 visited both England and Germany
- 30 visited both Italy and Germany
- 20 visited all three of these countries

How many went to England but not Italy or Germany?

We will learn how to solve puzzles like this in the second section of the chapter when counting the elements in a set is discussed.

1.1 Introduction to Sets

This section discusses operations on sets and the laws governing these set operations. These are fundamental notions that will be used throughout the remainder of this text. In the next two chapters we will see that probability and statistics are based on counting the elements in sets and manipulating set operations. Thus we first need to understand clearly the notion of sets and their operations.

✧ The Language of Sets

We begin here with some definitions of the language and notation used when working with sets. The most basic definition is "What is a set?" A **set** is a collection of items. These items are referred to as the **elements** or **members** of the set. For example, the set containing the numbers 1, 2, and 3 would be written $\{1,2,3\}$. Notice that the set is contained in curly brackets. This will help us distinguish sets from other mathematical objects.

When all the elements of the set are written out, we refer to this as **roster notation**. So the set containing the first 10 letters in the English alphabet would be written as $\{a,b,c,d,e,f,g,h,i,j\}$ in roster notation. If we wanted to refer to this set without writing all the elements, we could define the set in terms of its properties. This is called **set-builder notation**. So we write

$$\{x|x \text{ is one of the first 10 letters in the English alphabet}\}$$

This is read "the set of all x such that x is one of the first 10 letters in the English alphabet". If we will be using a set more than once in a discussion, it is useful to define the set with a symbol, usually an uppercase letter. So

$$S = \{a,b,c,d,e,f,g,h,i,j\}$$

We can say c is an element of the set $\{a,b,c,d,e,f,g,h,i,j\}$ or simply write $c \in S$. The symbol $\in$ is read "is an element of". We can also say that the set $R = \{c\}$ is a subset of our larger set S as every element in the set R is also in the set S.

> **Subsets**
> If every element of a set A is also an element of another set B, we say that A is a **subset** of B and write $A \subseteq B$. If A is not a subset of B, we write $A \nsubseteq B$.

Thus $\{1,2,4\} \subseteq \{1,2,3,4\}$, but $\{1,2,3,4\} \nsubseteq \{1,2,4\}$. Since every element in A is in A, we can write $A \subseteq A$. If there is a set B and every element in the set B is also in the set A but $B \neq A$, we say that B is a **proper subset** of A. This is written as $B \subset A$. Note the proper subset symbol $\subset$ is lacking the small horizontal line that the subset symbol $\subseteq$ has. The difference is rather like the difference between $<$ and $\leq$.

Some sets have no elements at all. We need some notation for this, simply leaving a blank space will not do!

> *HISTORICAL NOTE*
>
> **George Boole, 1815–1864**
>
> George Boole was born into a lower-class family in Lincoln, England, and had only a common school education. He was largely self-taught and managed to become an elementary school teacher. Up to this time any rule of algebra such as $a(x+y) = ax+ay$ was understood to apply only to numbers and magnitudes. Boole developed an "algebra" of sets where the elements of the sets could be not just numbers but anything. This then laid down the foundations for a fundamental way of thinking. Bertrand Russell, a great mathematician and philosopher of the 20th century, said that the greatest discovery of the 19th century was the nature of pure mathematics, which he asserted was discovered by George Boole. Boole's pamphlet "The Mathematical Analysis of Logic" maintained that the essential character of mathematics lies in its form rather than in its content. Thus mathematics is not merely the science of measurement and number but any study consisting of symbols and precise rules of operation. Boole founded not only a new algebra of sets but also a formal logic that we will discuss in Chapter L.

Empty Set
The **empty set**, written as $\emptyset$ or $\{\}$, is the set with no elements.

The empty set can be used to conveniently indicate that an equation has no solution. For example

$$\{x|x \text{ is real and } x^2 = -1\} = \emptyset$$

By the definition of subset, given any set A, we must have $\emptyset \subseteq A$.

EXAMPLE 1 **Finding Subsets** Find all the subsets of $\{a, b, c\}$.

Solution The subsets are

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$$ ✦

REMARK: Note that there are 8 subsets and 7 of them are proper subsets. In general, a set with n elements will have 2^n subsets. In the next chapter we will learn why this is so.

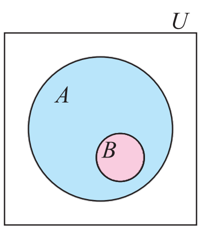


Figure 1.1

The empty set is the set with no elements. At the other extreme is the **universal set**. This set is the set of all elements being considered and is denoted by U. If, for example, we are to take a national survey of voter satisfaction with the president, the universal set is the set of all voters in this country. If the survey is to determine the effects of smoking on pregnant women, the universal set is the set of all pregnant women. The context of the problem under discussion will determine the universal set for that problem. The universal set must contain every element under discussion.

A **Venn diagram** is a way of visualizing sets. The universal set is represented by a rectangle and sets are represented as circles inside the universal set. For example, given a universal set U and a set A, Figure 1.1 is a Venn diagram that visualizes the concept that $A \subset U$. Figure 1.1 also visualizes the concept $B \subset A$. The U above the rectangle will be dropped in later diagrams as we will abide by the convention that the rectangle always represents the universal set.

✧ Set Operations

The first set operation we consider is the complement. The complement of set A are those members of set U that do **not** belong to A.

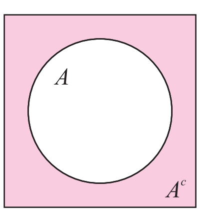


Figure 1.2
A^c is shaded.

Complement
Given a universal set U and a set $A \subset U$, the **complement of A**, written A^c, is the set of all elements that are in U but not in A, that is,

$$A^c = \{x|x \in U, \ x \notin A\}$$

A Venn diagram visualizing A^c is shown in Figure 1.2. Some alternate notations for the complement of a set are A' and $\overline{A}$.

EXAMPLE 2 **The Complements of Sets** Let $U = \{1,2,3,4,5,6,7,8,9\}$, $A = \{1,3,5,7,9\}$, $B = \{1,2,3,4,5\}$. Find A^c, B^c, U^c, $\emptyset^c$, and $(A^c)^c$ in roster notation.

Solution We have

$$\begin{aligned} A^c &= \{2,4,6,8\} \\ B^c &= \{6,7,8,9\} \\ U^c &= \emptyset \\ \emptyset^c &= \{1,2,3,4,5,6,7,8,9\} = U \\ (A^c)^c &= \{2,4,6,8\}^c \\ &= \{1,3,5,7,9\} = A \end{aligned}$$

✦

Note that in the example above we found $U^c = \emptyset$ and $\emptyset^c = U$. Additionally $(A^c)^c = A$. This can be seen using the Venn diagram in Figure 1.2, since the complement of A^c is all elements in U but not in A^c which is the set A. These three rules are called the **Complement Rules**.

Complement Rules

If U is a universal set, we must always have

$$U^c = \emptyset, \qquad \emptyset^c = U$$

If A is any subset of a universal set U, then

$$(A^c)^c = A$$

The next set operation is the union of two sets. This set includes the members of both sets A and B. That is, if an element belongs to set A or set B then it belongs to the union of A and B.

Set Union

The **union** of two sets A and B, written $A \cup B$, is the set of all elements that belong to A, or to B, or to both. Thus

$$A \cup B = \{x | x \in A \text{ or } x \in B \text{ or both}\}$$

REMARK: This usage of the word "or" is the same as in logic. It is the inclusive "or" where the elements that belong to both sets are part of the union. In English the use of "or" is often the exclusive "or". That is, if a meal you order at a restaurant comes with a dessert and you are offered cake or pie, you really only get one of the desserts. Choosing one dessert will exclude you from the other. If it was the logical "or" you could have both!

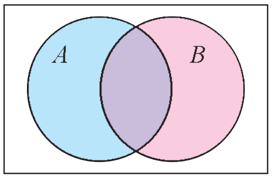


Figure 1.3
$A \cup B$ is shaded.

Our convention will be to drop the phrase "or both" but still maintain the same meaning. Note very carefully that this gives a particular definition to the word "or". Thus we will normally write

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

It can be helpful to say that the union of A and B, $A \cup B$, is all elements in A joined together with all elements in B. A Venn diagram visualizing this is shown in Figure 1.3 with the union shaded.

EXAMPLE 3 **The Union of Two Sets** Let $U = \{1,2,3,4,5,6\}$, $A = \{1,2,3,4\}$ and $B = \{1,4,5,6\}$. Find $A \cup B$ and $A \cup A^c$.

Solution We begin with the first set and join to it any elements in the second set that are not already there. Thus

$$\begin{aligned} A \cup B &= \{1,2,3,4\} \cup \{1,4,5,6\} \\ &= \{1,2,3,4,5,6\} \end{aligned}$$

Since $A^c = \{5,6\}$ we have

$$A \cup A^c = \{1,2,3,4\} \cup \{5,6\} = \{1,2,3,4,5,6\} = U$$

✦

The second result, $A \cup A^c = U$ is generally true. From Figure 1.2, we can see that if U is a universal set and $A \subset U$, then

$$A \cup A^c = U$$

Set Intersection

The **intersection** of two sets A and B, written $A \cap B$, is the set of all elements that belong to both the set A and to the set B. Thus

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

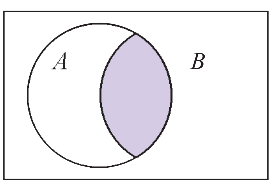


Figure 1.4

A Venn diagram is shown in Figure 1.4 with the intersection shaded.

EXAMPLE 4 **The Intersection of Two Sets** Find

a. $\{a,b,c,d\} \cap \{a,c,e\}$ **b.** $\{a,b\} \cap \{c,d\}$

Solution **a.** Only a and c are elements of both of the sets. Thus

$$\{a,b,c,d\} \cap \{a,c,e\} = \{a,c\}$$

b. The two sets $\{a,b\}$ and $\{c,d\}$ have no elements in common. Thus

$$\{a,b\} \cap \{c,d\} = \emptyset$$

✦

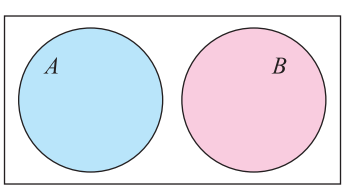


Figure 1.5
A and B are disjoint.

The sets $\{a,b\}$ and $\{c,d\}$ have no elements in common. These sets are called disjoint and can be visualized in Figure 1.5.

Disjoint Sets
Two sets A and B are **disjoint** if they have no elements in common, that is, if $A \cap B = \emptyset$.

An examination of Figure 1.2 or referring to the definition of A^c indicates that for any set A, A and A^c are disjoint. That is,

$$A \cap A^c = \emptyset$$

✧ Additional Laws for Sets

There are a number of laws for sets. They are referred to as commutative, associative, distributive, and De Morgan laws. We will consider two of these laws in the following examples.

HISTORICAL NOTE

Augustus De Morgan, 1806–1871

It was De Morgan who got George Boole interested in set theory and formal logic and then made significant advances upon Boole's epochal work. He discovered the De Morgan laws referred to in the last section. Boole and De Morgan are together considered the founders of the algebra of sets and of mathematical logic. De Morgan was a champion of religious and intellectual toleration and on several occasions resigned his professorships in protest of the abridgments of academic freedom of others.

EXAMPLE 5 **Establishing a De Morgan Law** Use a Venn diagram to show that

$$(A \cup B)^c = A^c \cap B^c$$

Solution We first consider the right side of this equation. Figure 1.6 shows a Venn diagram of A^c and B^c and $A^c \cap B^c$. We then notice from Figure 1.3 that this is $(A \cup B)^c$.

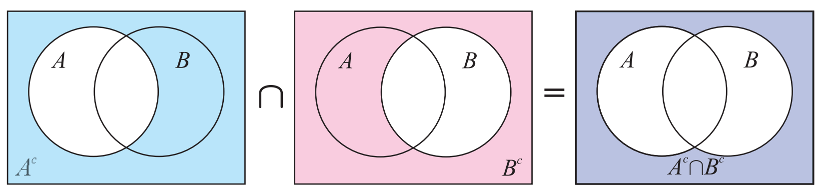


Figure 1.6 ✦

EXAMPLE 6 **Establishing the Distributive Law for Union** Use a Venn diagram to show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution Consider first the left side of this equation. In Figure 1.7a the sets A, $B \cap C$, and the union of these two are shown. Now for the right side of the equation refer to Figure 1.7b, where the sets $A \cup B$, $A \cup C$, and the intersection of these two sets are shown. We have the same set in both cases. ✦

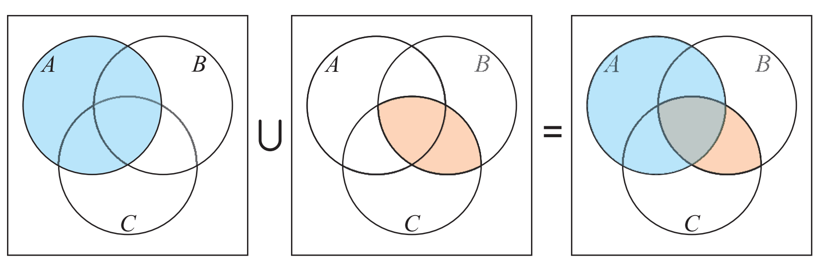


Figure 1.7a

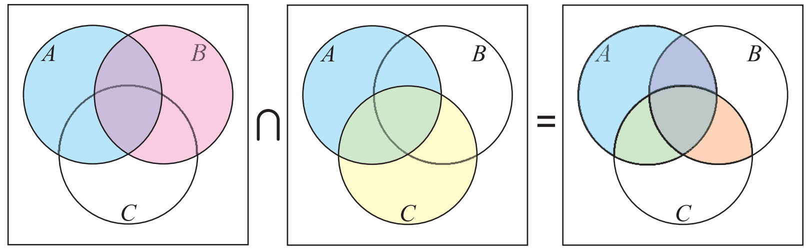


Figure 1.7b

We can summarize the laws we have found in the following list.

Laws for Set Operations

$A \cup B = B \cup A$	Commutative law for union
$A \cap B = B \cap A$	Commutative law for intersection
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative law for union
$A \cap (B \cap C) = (A \cap B) \cap C$	Associative law for intersection
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive law for union
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive law for intersection
$(A \cup B)^c = A^c \cap B^c$	De Morgan law
$(A \cap B)^c = A^c \cup B^c$	De Morgan law

✧ Applications

EXAMPLE 7 Using Set Operations to Write Expressions Let U be the universal set consisting of the set of all students taking classes at the University of Hawaii and

$B = \{x|x \text{ is currently taking a business course}\}$
$E = \{x|x \text{ is currently taking an English course}\}$
$M = \{x|x \text{ is currently taking a math course}\}$

Write an expression using set operations and show the region on a Venn diagram for each of the following:

a. The set of students at the University of Hawaii taking a course in at least one of the above three fields.

b. The set of all students at the University of Hawaii taking both an English course and a math course but not a business course.

c. The set of all students at the University of Hawaii taking a course in exactly one of the three fields above.

Solution

a. This is $B \cup E \cup M$. See Figure 1.8a.

b. This can be described as the set of students taking an English course (E) and also (intersection) a math course (M) and also (intersection) not a business course (B^c) or

$$E \cap M \cap B^c$$

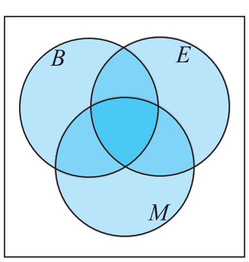


Figure 1.8a

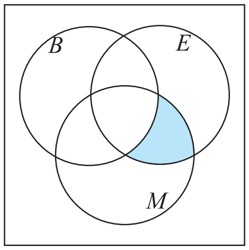


Figure 1.8b

This is the set of points in the universal set that are in both E and M but not in B and is shown in Figure 1.8b.

c. We describe this set as the set of students taking business but not taking English or math ($B \cap E^c \cap M^c$) together with (union) the set of students taking English but not business or math ($E \cap B^c \cap M^c$) together with (union) the set of students taking math but not business or English ($M \cap B^c \cap E^c$) or

$$(B \cap E^c \cap M^c) \cup (B^c \cap E \cap M^c) \cup (B^c \cap E^c \cap M)$$

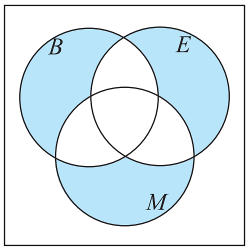


Figure 1.8c

This is the union of the three sets shown in Figure 1.8c. The first, $B \cap E^c \cap M^c$, consists of those points in B that are outside E and also outside M. The second set $E \cap B^c \cap M^c$ consists of those points in E that are outside B and M. The third set $M \cap B^c \cap E^c$ is the set of points in M that are outside B and E. The union of these three sets is then shown on the right in Figure 1.8c. ✦

REMARK: The word **only** means the same as exactly one. So a student taking only a business course would be written as $B \cap E^c \cap M^c$.

Self-Help Exercises 1.1

1. Let $U = \{1,2,3,4,5,6,7\}$, $A = \{l,2,3,4\}$, $B = \{3,4,5\}$, $C = \{2,3,4,5,6\}$. Find the following:

a. $A \cup B$ **b.** $A \cap B$
c. A^c **d.** $(A \cup B) \cap C$
e. $(A \cap B) \cup C$ **f.** $A^c \cup B \cup C$

2. Let U denote the set of all corporations in this country and P those that made profits during the last year, D those that paid a dividend during the last year, and L those that increased their labor force during the last year. Describe the following using the three sets P, D, L, and set operations. Show the regions in a Venn diagram.

a. Corporations in this country that had profits and also paid a dividend last year

b. Corporations in this country that either had profits or paid a dividend last year

c. Corporations in this country that did not have profits last year

d. Corporations in this country that had profits, paid a dividend, and did not increase their labor force last year

e. Corporations in this country that had profits or paid a dividend, and did not increase their labor force last year

1.1 Exercises

In Exercises 1 through 4, determine whether the statements are true or false.

1. a. $\emptyset \in A$ **b.** $A \in A$

2. a. $0 = \emptyset$ **b.** $\{x,y\} \in \{x,y,z\}$

3. a. $\{x|0 < x < -1\} = \emptyset$
b. $\{x|0 < x < -1\} = 0$

4. a. $\{x|x(x-1) = 0\} = \{0,1\}$
b. $\{x|x^2 + 1 < 0\} = \emptyset$

5. If $A = \{u,v,y,z\}$, determine whether the following statements are true or false.

a. $w \in A$ **b.** $x \notin A$
c. $\{u,x\} \cup A$ **d.** $\{y,z,v,u\} = A$

6. If $A = \{u, v, y, z\}$, determine whether the following statements are true or false.

a. $x \notin A$ **b.** $\{u, w\} \notin A$

c. $\{x, w\} \not\subset A$ **d.** $\emptyset \subset A$

7. List all the subsets of **a.** $\{3\}$, **b.** $\{3, 4\}$.

8. List all the subsets of **a.** $\emptyset$, **b.** $\{3, 4, 5\}$.

9. Use Venn diagrams to indicate the following.

a. $A \subset U$, $B \subset U$, $A \subset B^c$

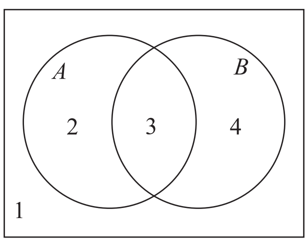
b. $A \subset U$, $B \subset U$, $B \subset A^c$

10. Use Venn diagrams to indicate the following.

a. $A \subset U$, $B \subset U$, $C \subset U$, $C \subset (A \cup B)^c$

b. $A \subset U$, $B \subset U$, $C \subset U$, $C \subset A \cap B$

For Exercises 11 through 14, indicate where the sets are located on the figure below and indicate if the sets found in part a and part b are disjoint or not.



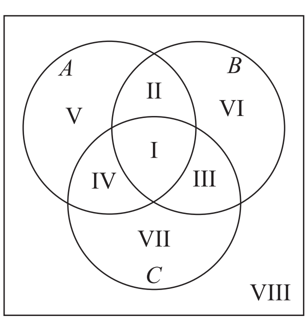
11. **a.** $A \cap B^c$ **b.** $A \cap B$

12. **a.** $A^c \cap B$ **b.** $A^c \cap B^c$

13. **a.** $A \cup B^c$ **b.** $(A \cup B)^c$

14. **a.** $A^c \cup B^c$ **b.** $(A \cap B)^c$

For Exercises 15 through 22, indicate where the sets are located on the figure below



15. **a.** $A \cap B \cap C$ **b.** $A \cap B^c \cap C^c$

16. **a.** $A \cap B \cap C^c$ **b.** $B \cap A^c \cap C^c$

17. **a.** $A^c \cap B^c \cap C^c$ **b.** $A \cap C \cap B^c$

18. **a.** $B \cap C \cap A^c$ **b.** $C \cap A^c \cap B^c$

19. **a.** $(A \cup B) \cap C^c$ **b.** $(A \cap B)^c \cap C$

20. **a.** $A \cup (B \cap C)$ **b.** $A \cup B \cup C^c$

21. **a.** $(A \cup B)^c \cap C$ **b.** $(A^c \cap B)^c \cup C$

22. **a.** $A \cup (B^c \cap C^c)$ **b.** $(A \cup B \cup C)^c \cap A$

In Exercises 23 through 30, find the indicated sets with

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

$A = \{1, 2, 3, 4, 5, 6\}$,

$B = \{4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 8, 9, 10\}$

23. **a.** $A \cap B$ **b.** $A \cup B$

24. **a.** A^c **b.** $A^c \cap B$

25. **a.** $A \cap B^c$ **b.** $A^c \cap B^c$

26. **a.** $A^c \cup B^c$ **b.** $(A^c \cup B^c)^c$

27. **a.** $A \cap B \cap C$ **b.** $(A \cap B \cap C)^c$

28. **a.** $A \cap (B \cup C)$ **b.** $A \cap (B^c \cup C)$

29. **a.** $A^c \cap B^c \cap C^c$ **b.** $(A \cup B \cup C)^c$

30. **a.** $A^c \cap B^c \cap C$ **b.** $A^c \cap B \cap C^c$

In Exercises 31 through 34, describe each of the sets in words.

Let U be the set of all residents of your state and let

$A = \{x | x \text{ owns an automobile}\}$

$H = \{x | x \text{ owns a house}\}$

31. **a.** A^c **b.** $A \cup H$ **c.** $A \cup H^c$

32. **a.** H^c **b.** $A \cap H$ **c.** $A^c \cap H$

33. **a.** $A \cap H^c$ **b.** $A^c \cap H^c$ **c.** $A^c \cup H^c$

34. **a.** $(A \cap H)^c$ **b.** $(A \cup H)^c$ **c.** $(A^c \cap H^c)^c$

In Exercises 35 through 38, let U, A, and H be as in the previous four problems, and let

$P = \{x|x \text{ owns a piano}\}$,

and describe each of the sets in words.

35. **a.** $A \cap H \cap P$ **b.** $A \cup H \cup P$
c. $(A \cap H) \cup P$

36. **a.** $(A \cup H) \cap P$ **b.** $(A \cup H) \cap P^c$
c. $A \cap H \cap P^c$

37. **a.** $(A \cap H)^c \cap P$ **b.** $A^c \cap H^c \cap P^c$
c. $(A \cup H)^c \cap P$

38. **a.** $(A \cup H \cup P)^c \cap A$ **b.** $(A \cup H \cup P)^c$
c. $(A \cap H \cap P)^c$

In Exercises 39 through 46, let U be the set of major league baseball players and let

$N = \{x|x \text{ plays for the New York Yankees}\}$
$S = \{x|x \text{ plays for the San Francisco Giants}\}$
$F = \{x|x \text{ is an outfielder}\}$
$H = \{x|x \text{ has hit 20 homers in one season}\}$

Write the set that represents the following descriptions.

39. **a.** Outfielders for the New York Yankees
b. New York Yankees who have never hit 20 homers in a season

40. **a.** San Francisco Giants who have hit 20 homers in a season.
b. San Francisco Giants who do not play outfield.

41. **a.** Major league ball players who play for the New York Yankees or the San Francisco Giants.
b. Major league ball players who play for neither the New York Yankees nor the San Francisco Giants.

42. **a.** San Francisco Giants who have never hit 20 homers in a season.
b. Major league ball players who have never hit 20 homers in a season.

43. **a.** New York Yankees or San Francisco Giants who have hit 20 homers in a season.
b. Outfielders for the New York Yankees who have never hit 20 homers in a season.

44. **a.** Outfielders for the New York Yankees or San Francisco Giants.
b. Outfielders for the New York Yankees who have hit 20 homers in a season.

45. **a.** Major league outfielders who have hit 20 homers in a season and do not play for the New York Yankees or the San Francisco Giants.
b. Major league outfielders who have never hit 20 homers in a season and do not play for the New York Yankees or the San Francisco Giants.

46. **a.** Major league players who do not play outfield, who have hit 20 homers in a season, and do not play for the New York Yankees or the San Francisco Giants.
b. Major league players who play outfield, who have never hit 20 homers in a season, and do not play for the New York Yankees or the San Francisco Giants.

In Exercises 47 through 52, let

$U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,2,3,4,5\}$,
$B = \{4,5,6,7\}$, $C = \{5,6,7,8,9,10\}$.

Verify that the identities are true for these sets.

47. $A \cup (B \cup C) = (A \cup B) \cup C$

48. $A \cap (B \cap C) = (A \cap B) \cap C$

49. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

50. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

51. $(A \cup B)^c = A^c \cap B^c$

52. $(A \cap B)^c = A^c \cup B^c$

Solutions to Self-Help Exercises 1.1

1. **a.** $A \cup B$ is the elements in A or B. Thus $A \cup B = \{1,2,3,4,5\}$.
b. $A \cap B$ is the elements in both A and B. Thus $A \cap B = \{3,4\}$.
c. A^c is the elements not in A (but in U). Thus $A^c = \{5,6,7\}$.

d. $(A \cup B) \cap C$ is those elements in $A \cup B$ and also in C. From **a** we have

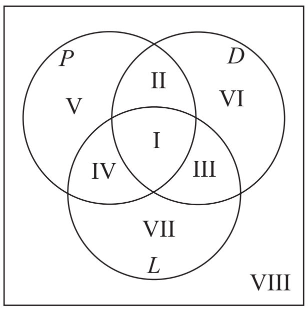
$$(A \cup B) \cap C = \{1,2,3,4,5\} \cap \{2,3,4,5,6\} = \{2,3,4,5\}$$

e. $(A \cap B) \cup C$ is those elements in $A \cap B$ or in C. Thus from **b**

$$(A \cap B) \cup C = \{3,4\} \cup \{2,3,4,5,6\} = \{2,3,4,5,6\}$$

f. $A^c \cup B \cup C$ is elements in B, or in C, or not in A. Thus

$$A^c \cup B \cup C = \{2,3,4,5,6,7\}$$



2. a. Corporations in this country that had profits and also paid a dividend last year is represented by $P \cap D$. This is regions I and II.

b. Corporations in this country that either had profits or paid a dividend last year is represented by $P \cup D$. This is regions I, II, III, IV, V, and VI.

c. Corporations in this country that did not have profits is represented by P^c. This is regions III, VI, VII, and VIII.

d. Corporations in this country that had profits, paid a dividend, and did not increase their labor force last year is represented by $P \cap D \cap L^c$. This is region II.

e. Corporations in this country that had profits or paid a dividend, and did not increase their labor force last year is represented by $(P \cup D) \cap L^c$. This is regions II, V, and VI.

1.2 The Number of Elements in a Set

APPLICATION
Breakfast Survey

In a survey of 120 adults, 55 said they had an egg for breakfast that morning, 40 said they had juice for breakfast, and 70 said they had an egg or juice for breakfast. How many had an egg but no juice for breakfast? How many had neither an egg nor juice for breakfast? See Example 1 for the answer.

✧ Counting the Elements of a Set

This section shows the relationship between the number of elements in $A \cup B$ and the number of elements in A, B, and $A \cap B$. This is our first counting principle. The examples and exercises in this section give some applications of this. In other applications we will count the number of elements in various sets to find probability.

The Notation $n(A)$
If A is a set with a finite number of elements, we denote the number of elements in A by $n(A)$.

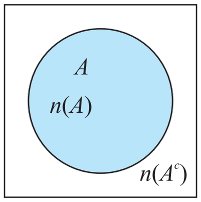


Figure 1.9

In Figure 1.9 we see the number $n(A)$ written inside the A circle and $n(A^c)$ written outside the set A. This indicates that there are $n(A)$ members in set A and $n(A^c)$ in set A^c. The number of elements in a set is also called the **cardinality** of the set.

There are two results that are rather apparent. First, the empty set $\emptyset$ has no elements $n(\emptyset) = 0$. For the second refer to Figure 1.10 where the two sets A and B are disjoint.

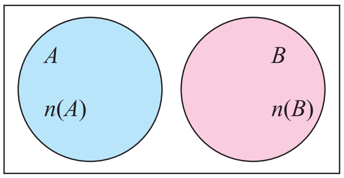


Figure 1.10

The Number in the Union of Disjoint Sets

If the sets A and B are disjoint, then

$$n(A \cup B) = n(A) + n(B)$$

A consequence of the last result is the following. In Figure 1.9, we are given a universal set U and a set $A \subset U$. Then since $A \cap A^c = \emptyset$ and $U = A \cup A^c$,

$$n(U) = n(A \cup A^c) = n(A) + n(A^c)$$

✧ Union Rule for Two Sets

Now consider the more general case shown in Figure 1.11. We assume that x is the number in the set A that are not in B, that is, $n(A \cap B^c)$. Next we have z, the number in the set B that are not in A, $n(A^c \cap B)$. Finally, y is the number in both A and B, $n(A \cap B)$ and w is the number of elements that are neither in A nor in B, $n(A^c \cap B^c)$. Then

$$\begin{aligned} n(A \cup B) &= x + y + z \\ &= (x + y) + (y + z) - y \\ &= n(A) + n(B) - n(A \cap B) \end{aligned}$$

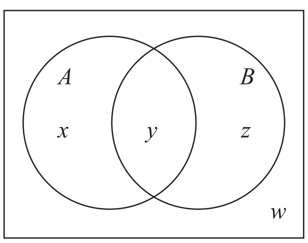


Figure 1.11

Alternatively, we can see that the total $n(A) + n(B)$ counts the number in the intersection $n(A \cap B)$ twice. Thus to obtain the number in the union $n(A \cup B)$, we must subtract $n(A \cap B)$ from $n(A) + n(B)$.

The Number in the Union of Two Sets

For any finite sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

EXAMPLE 1 **An Application of Counting** In a survey of 120 adults, 55 said they had an egg for breakfast that morning, 40 said they had juice for breakfast, and 70 said they had an egg or juice for breakfast. How many had an egg but no juice for breakfast? How many had neither an egg nor juice for breakfast?

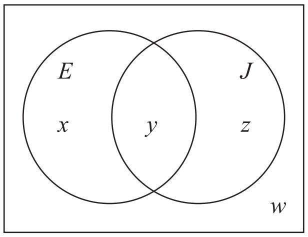


Figure 1.12a

Solution Let U be the universal set of adults surveyed, E the set that had an egg for breakfast, and J the set that had juice for breakfast. A Venn diagram is shown in Figure 1.12a. From the survey, we have that

$$n(E) = 55 \qquad n(J) = 40, \qquad n(E \cup J) = 70$$

Note that each of these is a sum. That is $n(E) = 55 = x + y$, $n(J) = 40 = y + z$ and $n(E \cup J) = 70 = x + y + z$. Since 120 people are in the universal set, $n(U) = 120 = x + y + z + w$.

The number that had an egg and juice for breakfast is given by $n(E \cap J)$ and is shown as the shaded region in Figure 1.12b. We apply the union rule:

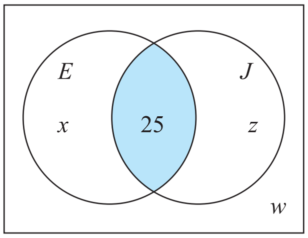


Figure 1.12b

$$\begin{aligned} n(E \cap J) &= n(E) + n(J) - n(E \cup J) \\ &= 55 + 40 - 70 \\ &= 25 \end{aligned}$$

We first place the number 25, just found, in the $E \cap J$ area in the Venn diagram in Figure 1.12b. Since the number of people who had eggs (with and without juice) is 55, then according to Figure 1.12b,

$$\begin{aligned} n(E) = 55 &= x + 25 \\ x &= 30 \end{aligned}$$

Similarly, the number who had juice (with and without an egg) is 40. Using Figure 1.12b,

$$\begin{aligned} n(J) = 40 &= z + 25 \\ z &= 15 \end{aligned}$$

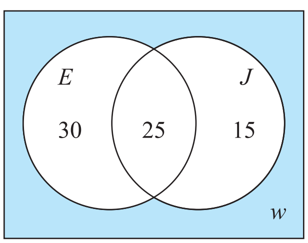


Figure 1.12c

These two results are shown in Figure 1.12c. We wish to find $w = n((E \cup J)^c)$. This is shown as the shaded region in Figure 1.12c. The unshaded region is $E \cup J$. We then have that

$$\begin{aligned} n(E \cup J) + n((E \cup J)^c) &= n(U) \\ n((E \cup J)^c) &= n(U) - n(E \cup J) \\ w &= 120 - 70 \\ w &= 50 \end{aligned}$$

And so there were 50 people in the surveyed group that had neither an egg nor juice for breakfast. ✦

✧ Counting With Three Sets

Many counting problems with sets have two sets in the universal set. We will also study applications with three sets in the universal set. The union rule for three sets is studied in the extensions for this section. In the example below, deductive reasoning is used to solve for the number of elements in each region of the Venn diagram. In cases where this will not solve the problem, systems of linear equations can be used to solve the Venn diagram. This is studied in the Chapter Project found in the Review section.

EXAMPLE 2 **European Travels** In a survey of 200 people that had just returned from a trip to Europe, the following information was gathered.

- 142 visited England
- 95 visited Italy
- 65 visited Germany
- 70 visited both England and Italy
- 50 visited both England and Germany
- 30 visited both Italy and Germany
- 20 visited all three of these countries

a. How many went to England but not Italy or Germany?
b. How many went to exactly one of these three countries?
c. How many went to none of these three countries?

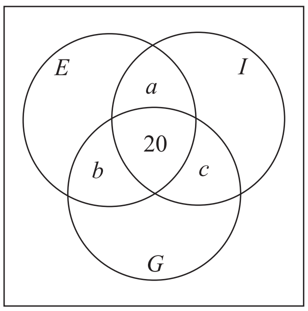


Figure 1.13a

Solution Let U be the set of 200 people that were surveyed and let
$E = \{x|x \text{ visited England}\}$
$I = \{x|x \text{ visited Italy}\}$
$G = \{x|x \text{ visited Germany}\}$

We first note that the last piece of information from the survey indicates that

$$n(E \cap I \cap G) = 20$$

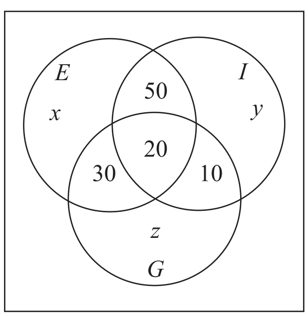


Figure 1.13b

Place this in the Venn diagram shown in Figure 1.13a. Recall that 70 visited both England and Italy, that is, $n(E \cap I) = 70$. If a is the number that visited England and Italy but not Germany, then, according to Figure 1.13a, $20 + a = n(E \cap I) = 70$. Thus $a = 50$. In the same way, if b is the number that visited England and Germany but not Italy, then $20 + b = n(E \cap G) = 50$. Thus $b = 30$. Also if c is the number that visited Italy and Germany but not England, then $20 + c = n(G \cap I) = 30$. Thus $c = 10$. All of this information is then shown in Figure 1.13b.

a. Let x denote the number that visited England but not Italy or Germany. Then, according to Figure 1.13b, $20 + 30 + 50 + x = n(E) = 142$. Thus $x = 42$, that is, the number that visited England but not Italy or Germany is 42.

b. Since $n(I) = 95$, the number that visited Italy but not England or Germany is given from Figure 1.13b by $95 - (50 + 20 + 10) = 15$. Since $n(G) = 65$, the number that visited Germany but not England or Italy is, according to Figure 1.13b, given by $65 - (30 + 20 + 10) = 5$. Thus, according to Figure 1.13c, the number who visited just one of the three countries is

$$42 + 15 + 5 = 62$$

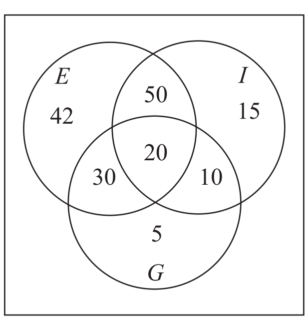


Figure 1.13c

c. There are 200 people in the U and so according to Figure 1.13c, the number that visited none of these three countries is given by

$$200 - (42 + 15 + 5 + 50 + 30 + 10 + 20) = 200 - 172 = 28$$ ✦

EXAMPLE 3 **Pizzas** At the end of the day the manager of Blue Baker wanted to know how many pizzas were sold. The only information he had is listed below. Use the information to determine how many pizzas were sold.

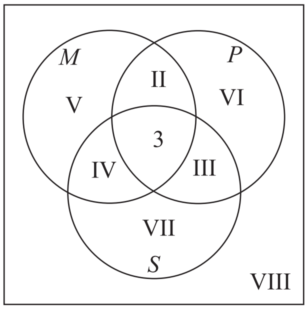


Figure 1.14a

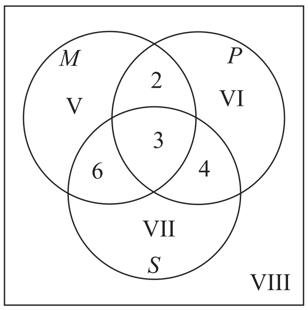


Figure 1.14b

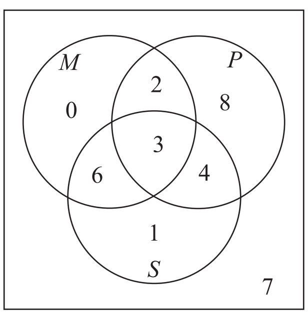


Figure 1.14c

- 3 pizzas had mushrooms, pepperoni, and sausage
- 7 pizzas had pepperoni and sausage
- 6 pizzas had mushrooms and sausage but not pepperoni
- 15 pizzas had two or more of these toppings
- 11 pizzas had mushrooms
- 8 pizzas had only pepperoni
- 24 pizzas had sausage or pepperoni
- 17 pizzas did not have sausage

Solution Begin by drawing a Venn diagram with a circle for pizzas that had mushrooms, a circle for pizzas that had pepperoni, and, pizzas that had sausage. In the center place a 3 since three pizzas had all these toppings. See Figure 1.14a.

Since 7 pizzas have pepperoni and sausage, 7 = 3 + III or III = 4. If 6 pizzas had mushrooms and sausage but not pepperoni, then IV = 6. The region for two or more of these toppings is 3 + II + III + IV = 15. Using III = 4 and IV = 6, that gives 3 + II + 4 + 6 = 15 or II = 2. This information is shown in Figure 1.14b.

Given that 11 pizzas had mushrooms, V + 2 + 3 + 6 = 11 and therefore V = 0. Since 8 pizzas had only pepperoni, VI = 8. With a total of 24 pizzas in the sausage or pepperoni region and knowing that VI = 8, we have 2 + 8 + 6 + 3 + 4 + VII = 24 or VII = 1. Finally, if 17 pizzas did not have sausage then 17 = V + 2 + VI + VIII = 0 + 2 + 8 + VIII. This gives VIII = 7 and our complete diagram is shown in Figure 1.14c.

To find the total number of pizzas sold, the 8 numbers in the completed Venn diagram are added:

$$0+2+8+6+3+4+1+7=31$$ ✦

Self-Help Exercises 1.2

1. Given that $n(A\cup B)=100$, $n(A\cap B^c)=50$, and $n(A\cap B)=20$, find $n(A^c\cap B)$.

2. The registrar reported that among 2000 students, 700 did not register for a math or English course, while 400 registered for both of these two courses. How many registered for exactly one of these courses?

3. One hundred shoppers are interviewed about the contents of their bags and the following results are found:

- 40 bought apple juice
- 19 bought cookies
- 13 bought broccoli
- 1 bought broccoli, apple juice, and cookies
- 11 bought cookies and apple juice
- 2 bought cookies and broccoli but not apple juice
- 24 bought only apple juice

Organize this information in a Venn diagram and find how many shoppers bought none of these items.

1.2 Exercises

1. If $n(A) = 100$, $n(B) = 75$, and $n(A \cap B) = 40$, what is $n(A \cup B)$?

2. If $n(A) = 200$, $n(B) = 100$, and $n(A \cup B) = 250$, what is $n(A \cap B)$?

3. If $n(A) = 100$, $n(A \cap B) = 20$, and $n(A \cup B) = 150$, what is $n(B)$?

4. If $n(B) = 100$, $n(A \cup B) = 175$, and $n(A \cap B) = 40$, what is $n(A)$?

5. If $n(A) = 100$ and $n(A \cap B) = 40$, what is $n(A \cap B^c)$?

6. If $n(U) = 200$ and $n(A \cup B) = 150$, what is $n(A^c \cap B^c)$?

7. If $n(A \cup B) = 500$, $n(A \cap B^c) = 200$, $n(A^c \cap B) = 150$, what is $n(A \cap B)$?

8. If $n(A \cap B) = 50$, $n(A \cap B^c) = 200$, $n(A^c \cap B) = 150$, what is $n(A \cup B)$?

9. If $n(A \cap B) = 150$ and $n(A \cap B \cap C) = 40$, what is $n(A \cap B \cap C^c)$?

10. If $n(A \cap C) = 100$ and $n(A \cap B \cap C) = 60$, what is $n(A \cap B^c \cap C)$?

11. If $n(A) = 200$ and $n(A \cap B \cap C) = 40$, $n(A \cap B \cap C^c) = 20$, $n(A \cap B^c \cap C) = 50$, what is $n(A \cap B^c \cap C^c)$?

12. If $n(B) = 200$ and $n(A \cap B \cap C) = 40$, $n(A \cap B \cap C^c) = 20$, $n(A^c \cap B \cap C) = 50$, what is $n(A^c \cap B \cap C^c)$?

For Exercises 13 through 20, let A, B, and C be sets in a universal set U. We are given $n(U) = 100$, $n(A) = 40$, $n(B) = 37$, $n(C) = 35$, $n(A \cap B) = 25$, $n(A \cap C) = 22$, $n(B \cap C) = 24$, and $n(A \cap B \cap C^c) = 10$. Find the following values.

13. $n(A \cap B \cap C)$

14. $n(A^c \cap B \cap C)$

15. $n(A \cap B^c \cap C)$

16. $n(A \cap B^c \cap C^c)$

17. $n(A^c \cap B \cap C^c)$

18. $n(A^c \cap B^c \cap C)$

19. $n(A \cup B \cup C)$

20. $n((A \cup B \cup C))^c$

Applications

21. Headache Medicine In a survey of 1200 households, 950 said they had aspirin in the house, 350 said they had acetaminophen, and 200 said they had both aspirin and acetaminophen.

a. How many in the survey had at least one of the two medications?

b. How many in the survey had aspirin but not acetaminophen?

c. How many in the survey had neither aspirin nor acetaminophen?

22. Newspaper Subscriptions In a survey of 1000 households, 600 said they received the morning paper but not the evening paper, 300 said they received both papers, and 100 said they received neither paper.

a. How many received the evening paper but not the morning paper?

b. How many received at least one of the papers?

23. Course Enrollments The registrar reported that among 1300 students, 700 students did not register for either a math or English course, 400 registered for an English course, and 300 registered for both types of courses.

a. How many registered for an English course but not a math course?

b. How many registered for a math course?

24. Pet Ownership In a survey of 500 people, a pet food manufacturer found that 200 owned a dog but not a cat, 150 owned a cat but not a dog, and 100 owned neither a dog or cat.

a. How many owned both a cat and a dog?

b. How many owned a dog?

25. Fast Food A survey by a fast-food chain of 1000 adults found that in the past month 500 had been to Burger King, 700 to McDonald's, 400 to Wendy's, 300 to Burger King and McDonald's, 250 to McDonald's and Wendy's, 220 to Burger King and Wendy's, and 100 to all three. How many went to

a. Wendy's but not the other two?

b. only one of them?

c. none of these three?

26. Investments A survey of 600 adults over age 50 found that 200 owned some stocks and real estate but no bonds, 220 owned some real estate and bonds but no stock, 60 owned real estate but no stocks or bonds, and 130 owned both stocks and bonds. How many owned none of the three?

27. Entertainment A survey of 500 adults found that 190 played golf, 200 skied, 95 played tennis, 100 played golf but did not ski or play tennis, 120 skied but did not play golf or tennis, 30 played golf and skied but did not play tennis, and 40 did all three.

a. How many played golf and tennis but did not ski?

b. How many played tennis but did not play golf or ski?

c. How many participated in at least one of the three sports?

28. Transportation A survey of 600 adults found that during the last year, 100 traveled by plane but not by train, 150 traveled by train but not by plane, 120 traveled by bus but not by train or plane, 100 traveled by both bus and plane, 40 traveled by all three, and 360 traveled by plane or train. How many did not travel by any of these three modes of transportation?

29. Magazines In a survey of 250 business executives, 40 said they did not read Money, Fortune, or Business Week, while 120 said they read exactly one of these three and 60 said they read exactly two of them. How many read all three?

30. Sales A furniture store held a sale that attracted 100 people to the store. Of these, 57 did not buy anything, 9 bought both a sofa and love seat, 8 bought both a sofa and chair, 7 bought both a love seat and chair. There were 24 sofas, 18 love seats, and 20 chairs sold. How many people bought all three items?

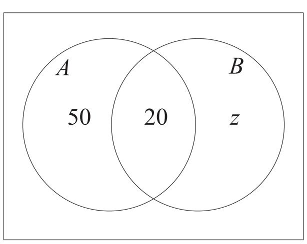
Extensions

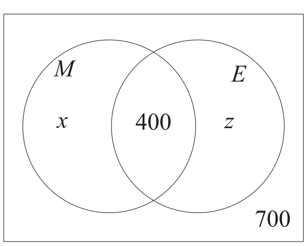
31. Use a Venn diagram to show that

$$\begin{aligned} n(A\cup B\cup C) = n(A)+n(B)+n(C)\\ -n(A\cap B)-n(A\cap C)\\ -n(B\cap C)+n(A\cap B\cap C) \end{aligned}$$

32. Give a proof of the formula in Exercise 31. Hint: Set $B\cup C = D$ and use union rule on $n(A\cup D)$. Now use the union rule two more times, recalling from the last section that $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.

Solutions to Self-Help Exercises 1.2





1. The accompanying Venn diagram indicates that

$$n(A\cap B^c) = 50, \quad n(A\cap B) = 20, \quad z = n(A^c\cap B)$$

Then, according to the diagram,

$$50+20+z = n(A\cup B) = 100$$

Thus $z = 30$.

2. The number of students that registered for exactly one of the courses is the number that registered for math but not English, $x = n(M\cap E^c)$, plus the number that registered for English but not math, $z = n(M^c\cap E)$. Then, according to the accompanying Venn diagram, $x+z+400+700 = 2000$. Thus $x+z = 900$. That is, 900 students registered for exactly one math or English course.

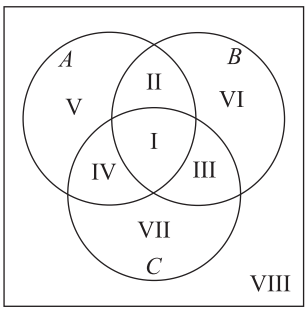
3. Let A be the set of shoppers who bought apple juice, B the set of shoppers who bought broccoli, and C the set of shoppers who bought cookies. This is shown in the first figure below. Since one shopper bought all three items, a 1 is placed in region I. Twenty-four shoppers bought only apple juice and this is region V.

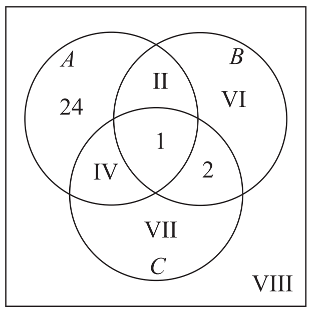
Given 2 shoppers bought cookies and broccoli but not apple juice, a 2 is placed in region III. This is shown in the next figure below.

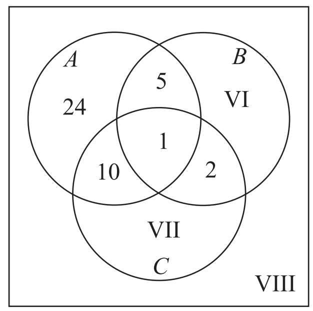
The statement "11 bought cookies and apple juice" includes those who bought broccoli and those who did not. We now know that one person bought all 3 items, so 11 − 1 = 10 people bought cookies and apple juice but not broccoli. A 10 is placed in region IV.

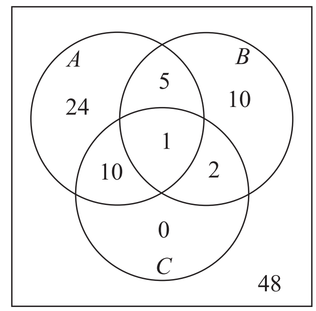
Now I + II + IV + V = 40 as we are told "40 bought apple juice." With the 10 in region IV we know 3 of the 4 values for set A and we can solve for region II: 40 = 24 + 10 + 1 + II gives II = 5. Place this in the Venn diagram as shown in the third figure below.

Examining the figure, we can use the total of 13 in the broccoli circle to solve for VI: 18 = 5 + 1 + 2 + VI gives VI = 10. The total of 19 in the cookies circle lets us solve for VII: 10 + 1 + 2 + VII = 13 gives VII = 0. The very last piece of information is that there were 100 shoppers. To solve for VIII we have 100 = 24 + 5 + 10 + 10 + 1 + 2 + 0 + VIII or VIII = 48. That is, 48 shoppers bought none of these items. The completed diagram is the final figure below.









1.3 Sample Spaces and Events

Many people have a good idea of the basics of probability. That is, if a fair coin is flipped, you have an equal chance of a head or a tail showing. However, as we proceed to study more advanced concepts in probability we need some formal definitions that will both agree with our intuitive understanding of probability and allow us to go deeper into topics such as conditional probability. This will tie closely to work we have done learning about sets.

✧ The Language of Probability

We begin the preliminaries by stating some definitions. It is very important to have a clear and precise language to discuss probability so pay close attention to the exact meanings of the terms below.

Experiments and Outcomes
An **experiment** is an activity that has observable results.
An **outcome** is the result of the experiment.

The following are some examples of experiments. Flip a coin and observe whether it falls "heads" or "tails." Throw a die (a small cube marked on each face with from one to six dots[1]) and observe the number of dots on the top face. Select a transistor from a bin and observe whether or not it is defective.

The following are some additional terms that are needed.

Sample Spaces and Trials
A **sample space** of an experiment is the set of all possible outcomes of the experiment. Each repetition of an experiment is called a **trial**.

For the experiment of throwing a die and observing the number of dots on the top face the sample space is the set

$$S = \{1, 2, 3, 4, 5, 6\}$$

In the experiment of flipping a coin and observing whether it falls heads or tails, the sample space is $S = \{\text{heads}, \text{tails}\}$ or simply $S = \{H, T\}$.

EXAMPLE 1 Determining the Sample Space An experiment consists of noting whether the price of the stock of the Ford Corporation rose, fell, or remained unchanged on the most recent day of trading. What is the sample space for this experiment?

Solution There are three possible outcomes depending on whether the price rose, fell, or remained unchanged. Thus the sample space S is

$$S = \{\text{rose}, \text{fell}, \text{unchanged}\}$$ ✦

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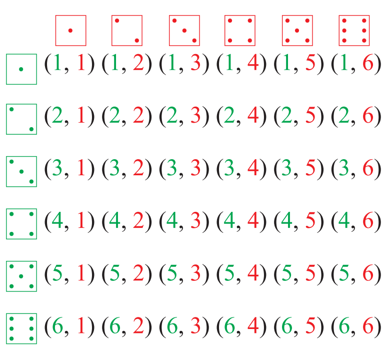

Figure 1.15

EXAMPLE 2 Determining the Sample Space Two dice, identical except that one is green and the other is red, are tossed and the number of dots on the top face of each is observed. What is the sample space for this experiment?

Solution Each die can take on its six different values with the other die also taking on all of its six different values. We can express the outcomes as order pairs. For example, (2, 3) will mean 2 dots on the top face of the green die and 3 dots on the top face of the red die. The sample space S is below. A more colorful version is shown in Figure 1.15.

$$\begin{aligned} S = \{&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),\\ &(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),\\ &(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),\\ &(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),\\ &(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),\\ &(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \end{aligned}$$ ✦

[1] There are also four-sided die, eight-sided die, and so on. However, the six-sided die is the most common and six-sided should be assumed when we refer to a die, unless otherwise specified.

If the experiment of tossing 2 dice consists of just observing the total number of dots on the top faces of the two dice, then the sample space would be

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

In short, the sample space depends on the precise statement of the experiment.

EXAMPLE 3 Determining the Sample Space A coin is flipped twice to observe whether heads or tails shows; order is important. What is the sample space for this experiment?

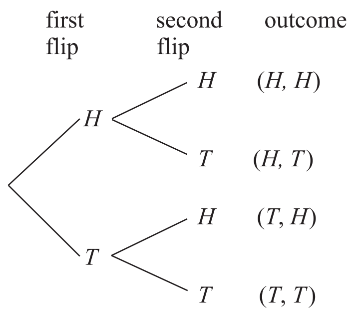

Figure 1.16

Solution The sample space S consists of the 4 outcomes

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$ ✦

✧ Tree Diagrams

In Example 3 we completed a task (flipped a coin) and then completed another task (flipped the coin again). In these cases, the experiment can be diagrammed with a tree. The **tree diagram** for Example 3 is shown in Figure 1.16. We see we have a first set of branches representing the first flip of the coin. From there we flip the coin again and have a second set of branches. Then trace along each branch to find the outcomes of the experiment. If the coin is tossed a third time, there will be eight outcomes.

EXAMPLE 4 Determining the Sample Space A die is rolled. If the die shows a 1 or a 6, a coin is tossed. What is the sample space for this experiment?

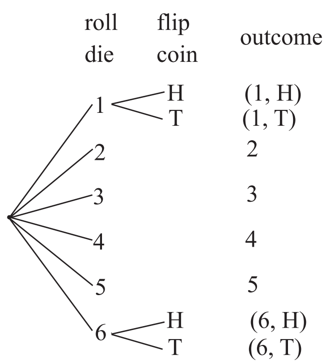

Figure 1.17

Solution Figure 1.17 shows the possibilities. We then have

$$S = \{(1,H), (1,T), 2, 3, 4, 5, (6,H), (6,T)\}$$ ✦

✧ Events

We start this subsection with the following definition of an event.

> **Events and Elementary Events**
> Given a sample space S for an experiment, an **event** is any subset E of S.
> An **elementary (or simple) event** is an event with a single outcome.

EXAMPLE 5 Finding Events Using the sample space from Example 3 find the events: "At least one head comes up" and "Exactly two tails come up." Are either events elementary events?

Solution "At least one head comes up" $= \{(H,H), (H,T), (T,H)\}$
"Exactly two tails come up" $= \{(T,T)\}$
The second event, "Exactly two tails come up" has only one outcome and so it is an elementary event. ✦

We can use our set language for union, intersection, and complement to describe events.

Union of Two Events
If E and F are two events, then $E \cup F$ is the union of the two events and consists of the set of outcomes that are in E or F.

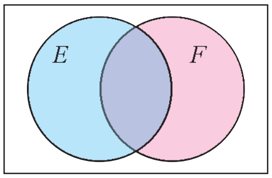

Figure 1.18

Thus the event $E \cup F$ is the event that "E or F occurs." Refer to Figure 1.18 where the event $E \cup F$ is the shaded region on the Venn diagram.

Intersection of Two Events
If E and F are two events, then $E \cap F$ is the intersection of the two events and consists of the set of outcomes that are in both E and F.

Thus the event $E \cap F$ is the event that "E and F both occur." Refer to Figure 1.18 where the event $E \cap F$ is the region where E and F overlap.

Complement of an Event
If E is an event, then E^c is the complement of E and consists of the set of outcomes that are not in E.

Thus the event E^c is the event that "E does not occur."

EXAMPLE 6 **Determining Union, Intersection, and Complement** Consider the sample space given in Example 2. Let E consist of those outcomes for which the number of dots on the top faces of both dice is 2 or 4. Let F be the event that the sum of the number of dots on the top faces of the two dice is 6. Let G be the event that the sum of the number of dots on the top faces of the two dice is less than 11.

a. List the elements of E and F.
b. Find $E \cup F$.
c. Find $E \cap F$.
d. Find G^c.

Solution
a. $E = \{(2,2),(2,4),(4,2),(4,4)\}$ and $F = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
b. $E \cup F = \{(2,2),(2,4),(4,2),(4,4),(1,5),(3,3),(5,1)\}$
c. $E \cap F = \{(2,4),(4,2)\}$
d. $G^c = \{(5,6),(6,5),(6,6)\}$ ✦

If S is a sample space, $\emptyset \subseteq S$, and thus $\emptyset$ is an event. We call the event $\emptyset$ the **impossible event** since the event $\emptyset$ means that no outcome has occurred, whereas, in any experiment some outcome *must* occur.

The Impossible Event
The empty set, $\emptyset$, is called the **impossible event**.

For example, if H is the event that a head shows on flipping a coin and T is the event that a tail shows, then $H \cap T = \emptyset$. The event $H \cap T$ means that both heads and tails shows, which is impossible.

Since $S \subseteq S$, S is itself an event. We call S the **certainty event** since any outcome of the experiment must be in S. For example, if a fair coin is flipped, the event $H \cup T$ is certain since a head or tail must occur.

The Certainty Event
Let S be a sample space. The event S is called the **certainty event**.

We also have the following definition for mutually exclusive events. See Figure 1.19.

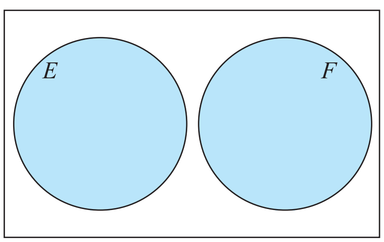

Figure 1.19

Mutually Exclusive Events
Two events E and F are said to be **mutually exclusive** if the sets are **disjoint**. That is,

$$E \cap F = \emptyset$$

Standard Deck of 52 Playing Cards
A standard deck of 52 playing cards has four 13-card suits: clubs ♣, diamonds ♢, hearts ♡, and spades ♠. The diamonds and hearts are red, while the clubs and spades are black. Each 13-card suit contains cards numbered from 2 to 10, a jack, a queen, a king, and an ace. The jack, queen, king, and ace can be considered respectively as number 11, 12, 13, and 14. In poker the ace can be either a 14 or a 1. See Figure 1.20.

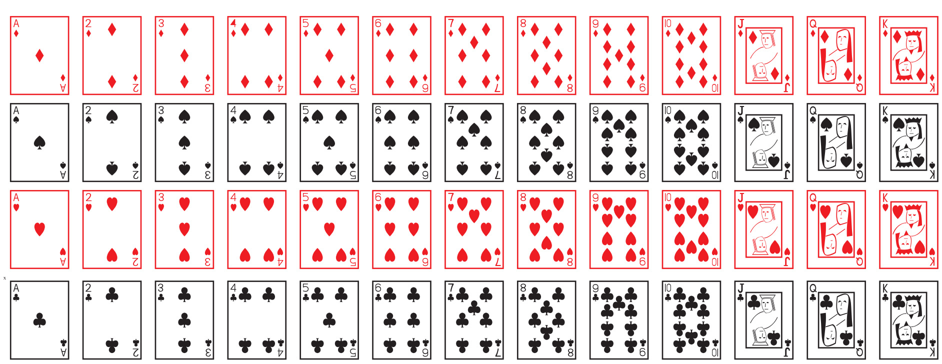

Figure 1.20

EXAMPLE 7 **Determining if Sets Are Mutually Exclusive** Let a card be chosen from a standard deck of 52 cards. Let E be the event consisting of drawing a 3. Let F be the event of drawing a heart. Let G be the event of drawing a Jack. Are E and F mutually exclusive? Are E and G?

Solution Since $E \cap F$ is the event that the card is a 3 and is a heart,

$$E \cap F = \{3\heartsuit\} \neq \emptyset$$

and so these events are not mutually exclusive. The event $E \cap G$ is the event that the card is a 3 and a jack, so

$$E \cap G = \emptyset$$

therefore E and G are mutually exclusive. ✦

✧ Continuous Sample Spaces

In all the previous examples we were able to list each outcome in the sample space, even if the list is rather long. But consider the outcomes of an experiment where the time spent running a race is measured. Depending on how the time is measured, an outcome could be 36 seconds or 36.0032 seconds. The values of the outcomes are not restricted to whole numbers and so the sample space must be described, rather than listed. In the case of the race we could say $S = \{t | t \geq 0, t \text{ in seconds}\}$. Then the event E that a person takes less than 35 seconds to run the race would be written $E = \{t | t < 35 \text{ seconds}\}$.

EXAMPLE 8 **Weighing Oranges** At a farmer's market there is a display of fresh oranges. The oranges are carefully weighed. What is a sample space for this experiment? Describe the event that a orange weighs 100 grams or more. Describe the event that a orange weighs between 200 and 250 grams.

Solution Since the weight of the orange can be any positive number,

$$S = \{w | w > 0, w \text{ in grams}\}$$

Note that $w = 0$ is not included as if the weight was zero there would be no orange! The event that an orange weighs 100 grams or more is

$$E = \{w | w \geq 100, w \text{ in grams}\}$$

Here note that we use $\geq$, not $>$ as the value of exactly 100 grams needs to be included. The event that the orange weighs between 200 and 250 grams is

$$F = \{w | 200 < w < 250, w \text{ in grams}\}$$

where strict inequalities are used as the weight is between those values. ✦

Self-Help Exercises 1.3

1. Two tetrahedrons (4 sided), each with equal sides numbered from 1 to 4, are identical except that one is red and the other green. If the two tetrahedrons are tossed and the number on the bottom face of each is observed, what is the sample space for this experiment?
2. Consider the sample space given in the previous exercise. Let E consist of those outcomes for which both (tetrahedron) dice show an odd number. Let F be the event that the sum of the two numbers on these dice is 5. Let G be the event that the sum of the two numbers is less than 7.

   a. List the elements of E and F.

   b. Find $E \cap F$.

   c. Find $E \cup F$.

   d. Find G^c.
3. A hospital carefully measures the length of every baby born. What is a sample space for this experiment? Describe the events

   a. the baby is longer than 22 inches.

   b. the baby is 20 inches or shorter.

   c. the baby is between 19.5 and 21 inches long.

1.3 Exercises

1. Let $S = \{a, b, c\}$ be a sample space. Find all the events.
2. Let the sample space be $S = \{a, b, c, d\}$. How many events are there?
3. A coin is flipped three times, and heads or tails is observed after each flip. What is the sample space? Indicate the outcomes in the event "at least 2 heads are observed."
4. A coin is flipped, and it is noted whether heads or tails show. A die is tossed, and the number on the top face is noted. What is the sample space of this experiment?
5. A coin is flipped three times. If heads show, one is written down. If tails show, zero is written down. What is the sample space for this experiment? Indicate the outcomes if "one is observed at least twice."
6. Two tetrahedrons (4 sided), each with equal sides numbered from 1 to 4, are identical except that one is red and the other green. If the two tetrahedrons are tossed and the number on the bottom face of each is observed, indicate the outcomes in the event "the sum of the numbers is 4."
7. An urn holds 10 identical balls except that 1 is white, 4 are black, and 5 are red. An experiment consists of selecting a ball from the urn and observing its color. What is a sample space for this experiment? Indicate the outcomes in the event "the ball is not white."
8. For the urn in Exercise 7, an experiment consists of selecting 2 balls in succession without replacement and observing the color of each of the balls. What is the sample space of this experiment? Indicate the outcomes of the event "no ball is white."
9. Ann, Bubba, Carlos, David, and Elvira are up for promotion. Their boss must select three people from this group of five to be promoted. What is the sample space? Indicate the outcomes of the event "Bubba is selected."
10. A restaurant offers six side dishes: rice, macaroni, potatoes, corn, broccoli, and carrots. A customer must select two different side dishes for his dinner. What is the sample space? List the outcomes of the event "Corn is selected."
11. An experiment consists of selecting a digit from the number 112964333 and observing it. What is a sample space for this experiment? Indicate the outcomes in the event that "an even digit."
12. An experiment consists of selecting a letter from the word CONNECTICUT and observing it. What is a

sample space for this experiment? Indicate the outcomes of the event “a vowel is selected.”

13. An inspector selects 10 transistors from the production line and notes how many are defective.

a. Determine the sample space.

b. Find the outcomes in the set corresponding to the event E “at least 6 are defective.”

c. Find the outcomes in the set corresponding to the event F “at most 4 are defective.”

d. Find the sets $E \cup F$, $E \cap F$, E^c, $E \cap F^c$, $E^c \cap F^c$.

e. Find all pairs of sets among the nonempty ones listed in part (d) that are mutually exclusive.

14. A survey indicates first whether a person is in the lower income group (L), middle income group (M), or upper income group (U), and second which of these groups the father of the person is in.

a. Determine the sample space using the letters L, M, and U.

b. Find the outcomes in the set corresponding to the event E “the person is in the lower income group.”

c. Find the outcomes in the set corresponding to the event F “the person is in the higher income group.”

d. Find the sets $E \cup F$, $E \cap F$, E^c, $E \cap F^c$, $E^c \cap F^c$.

e. Find all pairs of sets listed in part (d) that are mutually exclusive.

15. A corporate president decides that for each of the next three fiscal years success (S) will be declared if the earnings per share of the company go up at least 10% that year and failure (F) will occur is less than 10%.

a. Determine the sample space using the letters S and F.

b. Find the outcomes in the set corresponding to the event E “at least 2 of the next 3 years is a success.”

c. Find the outcomes in the set corresponding to the event G “the first year is a success.”

d. Find and describe the sets $E \cup G$, $E \cap G$, G^c, $E^c \cap G$, and $(E \cup G)^c$.

e. Find all pairs of sets listed in part (d) that are mutually exclusive.

16. Let E be the event that the life of a certain light bulb is at least 100 hours and F that the life is at most 200 hours. Describe the sets:

a. $E \cap F$ **b.** F^c **c.** $E^c \cap F$ **d.** $(E \cup F)^c$

17. Let E be the event that a pencil is 10 cm or longer and F the event that the pencil is less than 25 cm. Describe the sets:

a. $E \cap F$ **b.** E^c **c.** $E \cap F^c$ **d.** $(E \cup F)^c$

In Exercises 18 through 23, S is a sample space and E, F, and G are three events. Use the symbols $\cap$, $\cup$, and ^{c} to describe the given events.

18. F but not E

19. E but not F

20. Not F or not E

21. Not F and not E

22. Not F, nor E, nor G

23. E and F but not G

24. Let S be a sample space consisting of all the integers from 1 to 20 inclusive, E the first 10 of these, and F the last 5 of these. Find $E \cap F$, $E^c \cap F$, $(E \cup F)^c$, and $E^c \cap F^c$.

25. Let S be the 26 letters of the alphabet, E be the vowels $\{a, e, i, o, u\}$, F the remaining 21 letters, and G the first 5 letters of the alphabet. Find the events $E \cup F \cup G$, $E^c \cup F^c \cup G^c$, $E \cap F \cap G$, and $E \cup F^c \cup G$.

26. A bowl contains a penny, a nickel, and a dime. A single coin is chosen at random from the bowl. What is the sample space for this experiment? List the outcomes in the event that a penny or a nickel is chosen.

27. A cup contains four marbles. One red, one blue, one green, and one yellow. A single marble is drawn at random from the cup. What is the sample space for this experiment? List the outcomes in the event that a blue or a green marble is chosen.

Solutions to Self-Help Exercises 1.3

1. Consider the outcomes as ordered pairs, with the number on the bottom of the red one the first number and the number on the bottom of the white one the second number. The sample space is

$$\begin{aligned} S = \{ &(1,1),(1,2),(1,3),(1,4),\\ &(2,1),(2,2),(2,3),(2,4),\\ &(3,1),(3,2),(3,3),(3,4),\\ &(4,1),(4,2),(4,3),(4,4)\} \end{aligned}$$

2. **a.** $E = \{(1,1),(1,3),(3,1),(3,3)\}$, and $F = \{(1,4),(2,3),(3,2),(4,1)\}$

   b. $E \cap F = \emptyset$

   c. $E \cup F = \{(1,1),(1,3),(3,1),(3,3),(1,4),(2,3),(3,2),(4,1)\}$

   d. $G^c = \{(3,4),(4,3),(4,4)\}$

3. Since the baby can be any length greater than zero, the sample space is

$$S = \{x | x > 0, x \text{ in inches}\}$$

   a. $E = \{x | x > 22, x \text{ in inches}\}$

   b. $F = \{x | x \leq 20, x \text{ in inches}\}$

   c. $G = \{x | 19.5 < x < 21, x \text{ in inches}\}$

1.4 Basics of Probability

✧ Introduction to Probability

We first consider sample spaces for which the outcomes (elementary events) are equally likely. For example, a head or tail is equally likely to come up on a flip of a fair coin. Any of the six numbers on a fair die is equally likely to come up on a roll. We will refer to a sample space S whose individual elementary events are equally likely as a **uniform sample space**. We then give the following definition of the **probability** of any event in a uniform sample space.

Probability of an Event in a Uniform Sample Space

If S is a finite uniform sample space and E is any event, then the **probability of E**, $P(E)$, is given by

$$P(E) = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{n(E)}{n(S)}$$